7.1 Baryon Wave Functions and Magnetic Moments in the Constituent-Quark Model (7 pts.)

As discussed in class, the physical realizations of the baryon wave functions in the Constituent-Quark Model (CQM) correspond to fully symmetric flavor-spin parts (notation: use $u^\uparrow (s^\uparrow)$ for an up (strange) quark with spin up (down), etc.; identify the position of writing a quark in the baryon wave function with the particle label, so that the particle exchange operator for, e.g., quark 1 and 3, acts as follows: $P_{13}|u^\uparrow d^\uparrow s^\uparrow\rangle = |s^\uparrow d^\uparrow u^\uparrow\rangle$).

(a) The baryon decuplet is realized by combining the fully symmetric flavor and spin wave functions in the $(10; 4)$ representation. Starting from the wave function of the $(S_z^+ = +\frac{3}{2})$ state, construct the (normalized) wave functions of the remaining 3 states, as well as of the $(S_z^0 = +\frac{3}{2})$ and $(S_z^0 = 0)$ states, by using the flavor step operators, $\chi^I = \frac{1}{2}(\lambda_1 \pm i\lambda_2)$ and $\chi^U = \frac{1}{2}(\lambda_6 \pm i\lambda_7)$ for $I$- and $U$-spin, respectively.

(b) The baryon octet representation corresponds to $\frac{1}{\sqrt{2}}[(8; S_z^+ = +\frac{1}{2}) + (8; S_z^+ = +\frac{3}{2})]$, where “$S$” (“$A$”) denotes (anti-) symmetry with respect to the first 2 quarks. Construct the flavor-spin wave function of a $S_z^+ = +\frac{1}{2}$ proton. Start from the $1\leftrightarrow 2$ antisymmetric flavor part, $p_A = \frac{1}{\sqrt{2}}(ud - du)u$, and construct $p_S$ by requiring orthogonality to both $p_A$ and the decuplet $\Delta^+$; then combine $p_A$ and $p_S$ with the $1\leftrightarrow 2$ anti-/symmetric spin wave functions $\chi_A$ and $\chi_S$, respectively.

(c) Explicitly verify symmetry of the flavor-spin proton wave function constructed in (b) under quark exchange $P_{12}$, $P_{13}$ and $P_{23}$.

(d) In the CQM the magnetic moment operator for a hadron $h$ is defined by

$$\mu_h = \sum_i \mu_i \sigma^j_i \quad \mu_i = \frac{e_i}{2m_i}$$

where the sum is over all constituent quarks (charge $e_i$ and mass $m_i$) in the hadron. Compute the magnetic moment of the (spin-up) proton, $\mu_p$, in terms of the $u$- and $d$-quark ones, $\mu_{u,d}$. Obtain the neutron magnetic moment by the interchange $u \leftrightarrow d$, and compute $\mu_n/\mu_p$ assuming $m_u = m_d$. How does your result compare to the experimental value of -0.685?

7.2 Electron Scattering and Parton Model (3 pts.)

The elastic cross section for scattering an electron (with initial and final 4-momenta $(E, \vec{k})$ and $(E', \vec{k}')$, respectively) off a spin-$\frac{1}{2}$ point particle of mass $M$ can be written as

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4(\theta/2)} E' \left(\cos^2 \frac{\theta}{2} - \frac{q^2}{2M^2} \sin^2 \frac{\theta}{2}\right)$$

where $\theta = \angle(\vec{k}, \vec{k}')$ is the scattering angle (neglect the electron mass).
(a) Show that, for given incident energy in the lab system, the scattering angle is the only independent variable, by expressing $E'$ and the 4-momentum transfer $q^2$ in terms of $\theta$.

(b) For inelastic scattering, the outgoing proton breaks up and another variable (usually taken as $E'$ or $\nu = E - E'$) together with 2 structure functions, $W_{1,2}$, are needed. The differential cross section takes the form

$$
\frac{d\sigma}{d\Omega \, dE'} = \frac{\alpha^2}{4E'^2 \sin^4(\theta/2)} \left( W_2(\nu, q^2) \cos^2\frac{\theta}{2} + 2W_1(\nu, q^2) \sin^2\frac{\theta}{2} \right). \tag{3}
$$

Show that for $W_2(\nu, q^2) = \delta(\nu + q^2/2M)$ and $2W_1(\nu, q^2) = -q^2/(2M^2) \delta(\nu + q^2/2M)$, the elastic “point-like” cross section, Eq. (2), is recovered.

(c) Alternatively, introduce “new” point-like constituents with mass $m = xM$ by setting $W_2(\nu, q^2) = \delta(\nu + q^2/2m)$ and $2W_1(\nu, q^2) = -q^2/(2m^2) \delta(\nu + q^2/2m)$ in Eq. (3). Show that the redefined structure functions $F_1 \equiv MW_1$ and $F_2 \equiv \nu W_2$ lead to a cross section which only depends on the “mass fraction” $x$ of the “partons”, and not separately on $\nu$ and $q^2$. 