2.1 Medieval Castle Defense (2+3+3+2 pts.)

A castle is built on the hills of a river valley, \( h = 60 \text{m} \) above the river/valley level. The castle’s horizontal distance (from the bottom of the hill) to the nearside river bank is 180\( \text{m} \), and the river is another 55\( \text{m} \) wide. The castle is equipped with several cannons which can eject solid smooth rock spheres of 25\( \text{cm} \) diameter at a speed of \( v_0 = 45 \text{m/s} \) (the rock’s mass density is 2800\( \text{kg/m}^3 \)). The castle knights are about to fire the cannons in view of 500 enemy troops approaching the far-side river bank, but they have to figure out the appropriate launch angle, \( \theta_0 \). To save precious gunpowder, the court jester suggests to perform some estimates prior to the first shot.

(a) Analytical (benchmark) estimates: Neglecting air drag, derive the analytical expression for the horizontal range of the projectile as a function of \( \theta_0, v_0, h \) and \( g \) (= 9.8\( \text{m/s}^2 \)). Use your pocket calculator to obtain the projectile range, \( R(\theta_0) = x_{\text{max}}(\theta_0) \), for a few launch angles between 30\( \text{°} \) and 45\( \text{°} \) to roughly estimate the maximal theoretical range, \( R_{\text{max}} = R(\theta_{0\text{max}}) \), in vacuum. Sketch the trajectories in a hand-drawn graph (no need for accuracy except for \( \theta_0 \) and \( R \)).

(b) Write a FORTRAN code to compute the trajectory including a quadratic (in speed) air drag with a drag coefficient of 0.5 for the cannon ball (air density \( \rho_{\text{air}} = 1.29 \text{kg/m}^3 \)). Can the cannon ball reach the far-side river shore? Attach the source code and a plot of trajectories including the one with the maximal range, \( R_{\text{max}} \). Compute and plot the cannon ball’s speed upon impact, as function of \( \theta_0 \).

(c) How accurate does the launch angle have to be to hit a target of 2\( \text{m} \) horizontal size (neglect vertical size) which has just moved fully into the maximal range (i.e., its center is at \( R_{\text{max}} - 1\text{m} \))? How large is the angular variation to hit the target when it has moved to 50\% of the maximal range (evaluate both solutions, i.e., below and above \( \theta_{0\text{max}} \))? Use a linear interpolation technique in determining all impact locations.

(d) Suddenly a horizontal head wind of 20\( \text{mph} \) starts to blow, bringing also heavy rain fall; repeat part (b) for these conditions (approximate the effect of the rain drops by a 10\% increase of the air density).