6.1 Ideal Wave Propagation, Stability and Spectral Decomposition (cf. Ex. 6.1, 6.2, 6.4, 6.9, 6.12, 6.13) (3+2+2+3 pts.)

Write a FORTRAN program to numerically calculate the signal propagation on a uniform string (length \( L = 1 \text{ m} \), tension force \( F_T = 1200 \text{ N} \), total mass 30\( \text{ g} \), both ends fixed) using the ideal wave equation,

\[
\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2},
\]

for the displacement \( y(x, t) \) at string position \( x \) and time \( t \). In the discretized form of the equation, set the numerical coefficient \( r = \frac{c}{\Delta x/\Delta t} \) equal to one unless otherwise noted.

(a) Impart an initial Gaussian pluck on the string, \( y(x, t_0) = y_0 \exp\left[-(x-\bar{x})^2/2\sigma_x^2\right] \) with \( y_0 = 0.25\text{ m} \), \( \bar{x} = 0.3\text{ m} \) from the left end of the string and a full-width-at-half-maximum of 0.1\( \text{ m} \) (what is the corresponding value of \( \sigma_x \) ?). Study the signal propagation by plotting time snapshots of your solution, \( y(x, t_s) \), as the signal moves across the string. How are the signals reflected at the ends? Also produce a 3-D “surface” plot for \( y \) as a function of \( x \) and \( t \) (you can use, e.g., the splot command in gnuplot); the time axis should cover at least 1.2 times the period of the motion.

(b) Study and comment on the consequences of choosing values of \( r = 0.8, 1.2 \) in your numerical solution in (a).

(c) Perform a Fourier analysis of the string excitation in part (a), i.e., for a fixed position on the string (e.g. \( x_s = 0.5\text{ m} \)), record \( y(t) \) and evaluate the Fourier transform using a suitable time discretization. Plot the power spectrum vs. frequency and interpret the peaks in the spectrum in terms of the harmonics of the string. Repeat the Fourier analysis for a different position on the string (e.g. \( x_s = 0.7\text{ m} \)) and comment on the difference to \( x_s = 0.5\text{ m} \).

(d) Initialize the string with a more realistic “triangular” pluck, with a maximum \( y(0.3, t_0, 1) = 0.25\text{ m} \), which linearly connects to the ends (see Fig.1 below). Study its time dependence and power spectrum in analogy to parts (a) and (c) for \( x_s = 0.5\text{ m} \) only.

Fig.1: “Realistic” Pluck