FINAL EXAM

PHYS 625 (Fall 2012), 12/11/12

Name:

Signature:

Duration: 120 minutes
Show all your work for full/partial credit
Quote your answers in units of MeV *(or* GeV*) and* fm, *or combinations thereof*

No.	Points
1	
2	
3	
4	
5	
6	
Sum	

1.) *Conceptual Questions (no calculations required)* *(8+7 pts.)*

(A) The central nucleon-nucleon potential, $V_{12}^{\text{cent}}(r)$, is characterized by long-range attraction and a short-range repulsive core. When using the non-interacting Fermi-gas result for the nuclear wave function, Ψ_A, what problem arises in calculating the expectation value of the two-body potential contribution to the binding energy, $\langle \Psi_A | V(r) | \Psi_A \rangle$?
How is that problem solved in Brueckner-Bethe-Goldstone theory?

(B) Sketch the coordinate-space potential between a heavy quark and antiquark, $V_{Q\bar{Q}}(r)$, and give an approximate functional form. Briefly discuss the nature of the two main ingredients to this potential.

2.) *Empirical Nuclear Matter* *(18 pts.)*

Infinite nuclear matter is characterized by a saturation density of $\varrho_0 = 0.16\,\mathrm{fm}^{-3}$ and a binding energy per nucleon of $E_B/A = -16\,\mathrm{MeV}$.

(a) Calculate the Fermi momentum of nuclear matter [in MeV] by simply integrating up the nucleon momentum phase space at ϱ_0. Account for the spin-isospin degeneracy of nucleons.

(b) Calculate the non-relativistic nucleon Fermi energy, $\varepsilon_F = \varepsilon(k_F)$, and the average kinetic energy, $\langle \varepsilon_N \rangle$, of nucleons at $\varrho_0 = 0.16\,\mathrm{fm}^{-3}$. Quote both quantities in MeV.

(c) Using the kinetic energies of part (b), what average potential energy per nucleon, $\langle U \rangle$, is required to obtain the empirical binding energy?

3.) *Nuclear Shell Model* *(18 pts.)*

Consider nucleons moving in a collectively generated 3-D central harmonic oscillator potential,

$$U_c(r) = -U_0 + \frac{1}{2} M_N \omega^2 r^2 \tag{1}$$

with $U_0 = 50\,\text{MeV}$ and oscillator strength $\omega \simeq 15$ MeV for $A \simeq 60$. Its energy eigenvalues are

$$E_N = (N + \frac{3}{2})\hbar\omega - U_0 \tag{2}$$

with $N = 2(n-1) + l$, where $l \geq 0$ is the angular momentum quantum number and $n \geq 1$.

(a) Determine the proton shell fillings for N=0,1,2,3 and the corresponding (cumulative) "magic numbers" for shell closures.

(b) Now include a spin-orbit term in the potential: $U(r) = U_c(r) - 2\alpha \vec{L} \cdot \vec{S}$ with $\alpha \simeq 1$ MeV. Express the spin-orbit operator in terms of the eigenvalues j, l and s of $\vec{J}^2$, $\vec{L}^2$ and $\vec{S}^2$. (*hint*: $\vec{J} = \vec{L} + \vec{S}$).

(c) Evaluate $\vec{L} \cdot \vec{S}$ for the two possibilities $j = l \pm \frac{1}{2}$ and indicate how this can explain the experimentally observed magic numbers 2, 8, 20, 28.

4.) *Relativistic Mean-Field Model for Nuclear Matter* *(6+7+2 pts.)*

The Lagrangian for the relativistic σ-ω model is given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m_\sigma^2 \phi^2 - \frac{1}{4}(V_{\mu\nu})^2 + \frac{1}{2}m_\omega^2 V_\mu^2 + \bar{\psi}\left[i\partial\!\!\!/ - m_N - g_\omega V\!\!\!/ + g_\sigma \phi\right]\psi \ . \tag{3}$$

(a) Calculate the equation of motion for the scalar field.

(b) Apply the mean-field approximation for the meson fields, $\langle\phi\rangle \equiv \phi_0$, $\langle V_\nu\rangle \equiv \delta_{\nu 0} V_\nu$, and write down the Lagrangian in this approximation. Establish the relation between the scalar mean field and the scalar nucleon density, and identify an "in-medium" nucleon mass in terms of the scalar mean field.

(c) What is the physical effect of the vector mean-field on the nucleon energy?

5.) *Isospin Invariance of πN Lagrangian* *(16 pts.)*

A simple π-N-N interaction Lagrangian is given by

$$\mathcal{L}_{\pi NN} = g_{\pi NN}\ \bar{\psi}_N\, i\gamma_5\, \vec{\pi}\cdot\vec{\tau}\,\psi_N\ , \tag{4}$$

where arrows indicate vectors in isospin space (the nucleon spinors are doublets in this space).

(a) Show that the above Lagrangian is invariant under rotations in isospin space by applying an infinitesimal rotation to all field operators about an angle $\alpha \ll 1$,
$\psi_N \to (1 - i\vec{\alpha}\cdot\vec{\tau}/2)\psi_N\ , \quad \pi \to (1 + \vec{\alpha}\,\times)\,\vec{\pi}\ \ ,$
and verifying the invariance to leading order in α.

(b) Show that the above Lagrangian predicts relations between the physical couplings as

$$g_{pp\pi^0} = -g_{nn\pi^0} = \frac{1}{\sqrt{2}} g_{pn\pi^+} = \frac{1}{\sqrt{2}} g_{pn\pi^-} \tag{5}$$

where the physical (charged and neutral) pion fields are related to the cartesian ones, $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$, as $\pi^\pm = (\pi_1 \pm i\pi_2)/\sqrt{2}$ and $\pi^0 = \pi_3$.

6.) *Nambu Jona-Lasinio (NJL) Model for Hadron Structure* *(18 pts.)*

In mean-field approximation, the free energy density of the NJL model at finite temperature, T, and quark chemical potential, μ_q, is given by

$$\frac{\Omega(\mu_q, T; \chi_0)}{V} = G\chi_0^2 - d_q \int \frac{d^3p}{(2\pi)^3} \omega_p^* - T d_q \int \frac{d^3p}{(2\pi)^3} [\ln(1 + \mathrm{e}^{-(\omega_p^* - \mu_q)/T}) + \ln(1 + \mathrm{e}^{-(\omega_p^* + \mu_q)/T})] \quad (6)$$

with $d_q = N_s N_f N_c$: quark degeneracy, $\omega_p^* = (p^2 + m_q^{*2})^{1/2}$: quark energy, G: 4-quark coupling, $\chi_0 = \langle 0|\bar{q}q|0\rangle$: quark-antiquark condensate, $m_q^* = m_q - 2G\chi_0$: constituent quark mass.

(a) Take the $T \to 0$ limit of Eq. (6) and use the quark density in writing your final result. (*hint*: start by evaluating $T \ln(1 + \mathrm{e}^{-x/T})$ for $T \to 0$ for positive/negative x)

(b) Take the limit of vanishing chemical potential of your result from part (a) to show that

$$\frac{\Omega(m_q^*)}{V} = \frac{(m_q^* - m_q)^2}{4G} - d_q \int \frac{d^3p}{(2\pi)^3} \omega_p^* \, . \quad (7)$$

Briefly interpret the physical origin of the two terms.

(c) Derive the self-consistency gap equation for the constituent quark mass. What additional ingredient is necessary to render the equation meaningful, and what is the pertinent physical interpretation of this ingredient?

Formula Sheet

Euler Lagrange equation for field X:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu X)} \right) - \frac{\partial \mathcal{L}}{\partial X} = 0$$

Commutation relations

$$[\tau_i, \tau_j] = 2i\varepsilon_{ijk}\tau_k$$

Hadron masses:

$$m_N = 940\,\mathrm{MeV},\; m_\pi = 140\,\mathrm{MeV}$$

Conversion factor: $\hbar c = 197\ \mathrm{MeV\,fm} = 0.197\ \mathrm{GeV\,fm}$