FINAL EXAM

PHYS 625 (Fall 2008), 12/10/08

Name:

Signature:

Duration: 120 minutes
Show all your work for full/partial credit
Quote your answers in units of MeV *(or* GeV*) and* fm, *or combinations thereof*

No.	Points
1	
2	
3	
4	
5	
Sum	

1.) *Fermi Gas Model of Nuclear Matter* (6+10+6 pts.)

For a convenient calculation of expectation values of 1- and 2-body operators, one defines pertinent 1- and 2-body density matrices,

$$\rho_{fi}^{(1)}(\mathbf{\vec{r}},\mathbf{\vec{r}'}) = \int d^3r_2\cdots d^3r_A\ \Psi_f^*(\mathbf{\vec{r}},\vec{r}_2,\ldots,\vec{r}_A)\ \Psi_i(\mathbf{\vec{r}'},\vec{r}_2,\ldots,\vec{r}_A) \tag{1}$$

$$\rho_{fi}^{(2)}(\mathbf{\vec{r}_1},\mathbf{\vec{r}_2};\mathbf{\vec{r}'_1},\mathbf{\vec{r}'_2}) = \int d^3r_3\cdots d^3r_A\ \Psi_f^*(\mathbf{\vec{r}_1},\mathbf{\vec{r}_2},\vec{r}_3,\ldots,\vec{r}_A)\ \Psi_i(\mathbf{\vec{r}'_1},\mathbf{\vec{r}'_2},\vec{r}_3,\ldots,\vec{r}_A) \tag{2}$$

in terms of A-body wave functions Ψ. In the non-interacting Fermi Gas model (FGM), the (fully antisymmetrized and normalized) A-nucleon wave function is given by a Slater determinant as

$$\Phi_{\vec{k}_1,\ldots,\vec{k}_A}(\vec{r}_1,\ldots,\vec{r}_A) = \frac{1}{\sqrt{A!}}\begin{vmatrix} \phi_{\vec{k}_1}(\vec{r}_1) & \cdots & \phi_{\vec{k}_A}(\vec{r}_1) \\ \vdots & \ddots & \vdots \\ \phi_{\vec{k}_1}(\vec{r}_A) & \cdots & \phi_{\vec{k}_A}(\vec{r}_A) \end{vmatrix} \tag{3}$$

where $\phi_{\vec{k}}(\vec{r}) = \mathrm{e}^{i\vec{k}\cdot\vec{r}}/\sqrt{V}$ are plane wave single-nucleon wave functions (spin and isospin quantum numbers are neglected here).

(a) Show that the 1-body density matrix for a local operator in the FGM is given by

$$\rho_{\mathrm{FG}}^{(1)}(\vec{r},\vec{r}) = \frac{1}{V}\ . \tag{4}$$

(b) Calculate the the 2-body density matrix in the FGM for local 2-body operators expressing it in the form

$$\rho_{\mathrm{FG}}^{(2)}(\vec{r}_1,\vec{r}_2;\vec{r}_1,\vec{r}_2) = \frac{g_-(x)}{V^2}\ . \tag{5}$$

with a suitably defined function $g_-(x)$ (taking the continuum limit).

(c) How does your results of part (b) change if the coordinate-space A-body wave function is combined with a fully antisymmetric spin-isospin wave function?

2.) *Relativistic Mean-Field Model for Nuclear Matter* *(16 pts.)*

The Lagrangian for the relativistic σ-ω model is given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m_\sigma^2 \phi^2 - \frac{1}{4}(V_{\mu\nu})^2 + \frac{1}{2} m_\omega^2 V_\mu^2 + \bar{\psi}\left[i\partial\!\!\!/ - m_N - g_\omega V\!\!\!/ + g_\sigma \phi\right]\psi \tag{6}$$

(a) Calculate the equation of motion for scalar field.

(b) Apply the mean-field approximation for the meson fields, $\langle\phi\rangle \equiv \phi_0$, $\langle V_\nu\rangle \equiv \delta_{\nu 0} V_\nu$, and write down the Lagrangian in this approximation. Establish the relation between the scalar mean field and the scalar nucleon density, and identify an "in-medium" nucleon mass in terms of the scalar mean field.

3.) Ω^- *Production in Hadronic Collisions* *(14 pts.)*
High-energy negatively charged pions (π^-) are directed on a hydrogen target (at rest). You want to find the minimum total pion energy necessary to produce the triple-strange baryon, Ω^- (=(sss) in the quark model). The reaction will be of the type

$$\pi^- + p \to \Omega^- + X \tag{7}$$

but you have to conserve baryon number, strangeness and electric charge.

(a) How many K-mesons (=($\bar{s}q$) with q=u,d) and of which type (K^+, K^0) do you have to minimally account for in the final state "X"?

(b) Calculate the minimal pion energy to produce the Ω^-.

4.) *Constituent Quark Model* *(21 pts.)*

Consider the $S = 3/2$ baryon decuplet; use the notation e.g. $d\uparrow$ for a spin-up *down* quark.

(a) Write down the spin-flavor wave function of the $\Delta^{++}(S_z = +3/2)$. How can it be compatible with the Pauli exclusion principle?

(b) Using V-spin step-down operators (changing u into s quarks) to calculate the (normalized) spin-flavor wave function of the $\Xi^{*,0}(S_z = +3/2)$ baryon which carries 2 strange quarks. By how much would you estimate the $\Xi^{*,0}$ mass to be larger then the Δ^{++} mass?

(c) Calculate the magnetic moments of the $\Delta^{++}(S_z = +3/2)$ and $\Xi^{*,0}(S_z = +3/2)$ using the hadronic operator

$$\mu_h = \sum_i \mu_i \, \sigma_z^i \quad , \quad \mu_i = \frac{e_i}{2m_i} \tag{8}$$

where the sum is over the quark constituents in the hadron and $\sigma_z^i = diag(1, -1)$ is the quark spin (Pauli) matrix (use $m_{u,d}$=0.35 GeV, m_s=0.5 GeV, electron charge e=0.3).

5.) *QCD Vacuum and Chiral Symmetry* *(10+14+3 pts.)*

(a) In the MIT bag model, the QCD vacuum is modeled by a background field generating pressure $P = -B$ and energy density $\epsilon = B$ (B is defined as negative). Hadrons are constructed as "bags" of empty vacuum stabilized by the kinetic energy of the quarks inside. Assume a spherical bag for which the ground state energy of one quark is given by $E_{kin} \sim 2.04/R$.

(a1) Write down the the total energy of the bag and find the radius, $R_{min}(B, N_q)$, which minimizes this energy (N_q: no. of quarks in the bag).

(a2) Equate the minimum energy to the proton mass and find the explicit values for bag radius (in fm) and bag constant, B (in GeV/fm^3).

(b) The effective quark Lagrangian of the Nambu-Jona-Lasinio (NJL) model is given by

$$\mathcal{L}_{\text{eff}} = \bar{q}(i\partial\!\!\!/ - m_q) + G\left[(\bar{q}q)^2 + (\bar{q}\gamma_5\vec{\tau}q)^2\right] \tag{9}$$

In the following, you can put the bare quark mass m_q=0 and ignore the pion-like interaction term (with γ_5).

(b1) Assume the presence of a nonzero mean-field quark condensate, $\chi_0 = \langle\bar{q}q\rangle$, and linearize the scalar interaction term to find the mean-field Lagrangian of the NJL model. Identify an effective quark mass in terms of the quark condensate χ_0.

(b2) The vacuum free energy following from (b1) takes the form

$$\Omega(\chi_0) = G\chi_0^2 - 2N_cN_f \int_0^\Lambda \frac{d^3p}{(2\pi)^3}\omega_p^* \tag{10}$$

with $\omega_p^* = (p^2 + (m_q^*)^2)^{1/2}$, N_c=3. Fix the parameters of the model at $\Lambda = 0.6$ GeV, $G\Lambda^2$=2.4, resulting in a quark condensate of $\chi_0 = 2(-0.24 \text{ GeV})^3$ (including u and d flavors, i.e., N_f=2). Compute the constituent quark mass and the bag constant defined as $B = \Omega(\chi_0) - \Omega(\chi_0 = 0)$. Compare the latter to the MIT bag model result.

(c) What is special about the pion in QCD?

Formula Sheet

Continuum limit of momentum summation (Fermi sea):

$$\sum_{\vec{k}_i} \rightarrow V \int_0^{k_F} \frac{d^3k}{(2\pi)^3} \quad , \quad \int_0^{k_F} \frac{d^3k}{(2\pi)^3} \mathrm{e}^{i\vec{k}\cdot\vec{r}} = \varrho \frac{3}{x^2}(\frac{1}{x}\sin x - \cos x) \ , \ \text{with: } x \equiv k_F r \ , \ \varrho = \frac{k_F^3}{6\pi^2}$$

Euler Lagrange equation for field X:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu X)} \right) - \frac{\partial \mathcal{L}}{\partial X} = 0$$

Total center-of-mass energy squared of n-particle state (Lorentz invariant):

$$s = (p_1 + p_2 + \cdots + p_n)^2 \quad , \quad p_i : \ 4 - \text{momenta}$$

Hadron masses:

$$m_N = 940\,\text{MeV}, \ m_\pi = 140\,\text{MeV}, \ m_\Omega = 1670\,\text{MeV}, \ m_K = 495\,\text{MeV}$$

Indefinite integral with $X = x^2 + a^2$:

$$\int dx\, x^2 \sqrt{X} = \frac{x}{4} X^{3/2} - \frac{a^2}{8}[x\sqrt{X} + a^2 \ln(x + \sqrt{X})] + \text{const}$$

Conversion factor: $\hbar c = 197\ \text{MeV fm} = 0.197\ \text{GeV fm}$