FINAL EXAM

PHYS 401 (Spring 2012), 05/09/12

Name:

Signature:

Duration: 120 minutes

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No.	Points
1	
2	
3	
4	
5	
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7	
Sum	

1.) *Multiple Choice* *(12 pts.)*

For each statement below, check the correct answer (no reasoning required).

(a) In the chaotic regime, the Lyapunov exponent, λ, defined for 2 independent trajectories of the same system starting close by, $|x_2(t) - x_1(t)| \propto \exp(\lambda t)$, is

- ☐ negative.
- ☐ zero.
- ☐ positive.
- ☐ complex.

(b) For a numerical Fourier analysis of a signal in the time domain with a sampling interval Δt, the Nyquist theorem states that the highest frequency that can be identified is

- ☐ $2/\Delta t$
- ☐ $1.5/\Delta t$
- ☐ $1/\Delta t$
- ☐ $1/(2\Delta t)$

(c) Compared to an unrestricted random walk, the Flory exponent of a self-avoiding random walk (in the same number of dimensions) is

- ☐ zero.
- ☐ smaller.
- ☐ equal.
- ☐ larger.

(d) In the 2D Ising model at zero external magnetic field, the full numerical treatment gives a critical exponent, β, for the magnetization as a function of temperature is

- ☐ zero.
- ☐ smaller
- ☐ equal
- ☐ larger

than in the mean-field approximation.

2.) *Realistic Drag Force* *(20 pts.)*

Consider a sports car capable of providing a maximal sustained power, P_{in}, on a horizontal road on a calm day (no wind; neglect tire-road fricton losses). The car has a total mass m, an effective cross sectional area A and a drag coefficient C, resulting in an air-drag force $F_{\text{drag}} = -B_2 v^2$ where $B_2 = \frac{1}{2} C \rho_{\text{air}} A$ (ρ_{air}: mass density of air).

(a) For C=0, use $F = ma$ and the definition of power, $P = dE/dt$, to rewrite the net acceleration as a function of P. Solve the equation for constant $P = P_{\text{in}}$ for given initial speed, v_0, to find the speed as a function of time, $v(t)$.

(b) Include the drag effect in the acceleration and use a first-order difference method to discretize the resulting equation. Express the speed at time step $i+1$ in terms of the previous time step i.

(c) Derive the terminal speed of the car and evaluate it for $P_{\text{in}} = 250hp$ ($1hp$=$0.745kW$), $A = 0.5m^2$, $C = 0.3$ and $\rho_{\text{air}} = 1.3kg/m^3$.

(d) Why is the solution in part (b) unrealistic for small v and how can this be improved?

3.) *Logistic Map* *(14 pts.)*

Consider the logistic map defined by the iterative equation

$$x_{n+1} = \mu \; x_n \; (1 - x_n) \tag{1}$$

where n denotes the time step and μ is the "growth" parameter.

(a) Find the first-order fixpoint, x^*, defined by $x = f(x)$ and determine the range of μ values for which the fixpoint is stable. (*hint*: consider small deviations from the fixpoint, $f(x^* + \Delta x) \simeq f(x^*) + f'(x^*)\Delta x$; what condition should the derivative $f'(x^*)$ satisfy to render the fixpoint stable?)

(b) The definition for second-order fixpoints is given by $x = f[f(x)]$. For μ=3.3 it has the solutions $x^* = 0.479, 0.700, 0.824$. Sketch both sides of the equation, $x = f[f(x)]$, graphically vs. x and determine whether the fixpoints are stable or unstable.

4.) *1-Dimensional Random Walk* *(12 pts.)*

Consider the most basic random walk in one dimension, where the walker starts at the origin and in each step moves backward or forward by one unit with equal probability of 50%. After n steps, the position of the walker can thus be expressed as

$$x_n = \sum_{i=1}^{n} s_i \quad , \qquad s_i = \pm 1 \text{ randomly} \ . \tag{2}$$

In the following, average quantities at given step n are defined as averaging over many (m) independent random walks,

$$\langle f(x_n) \rangle \equiv \frac{1}{m} \sum_{j=1}^{m} f(x_n) \ . \tag{3}$$

(a) Evaluate analytically the average position, $\langle x_n \rangle$, and variance, $\langle (x_n - \langle x_n \rangle)^2 \rangle$, of the 1-D random walker.

(b) The Flory exponent, ν, for the time (t) dependence of a diffusion process is defined as

$$\sqrt{\langle x^2 \rangle} \equiv A t^{\nu} \ . \tag{4}$$

Identifying the step-no. n with time, what is the Flory exponent for the 1D walker? What is the Flory exponent for a deterministic walker with constant velocity?

5.) *Koch Curve* *(10 pts.)*

The fractal dimension of a curve, d_f, is defined by the dependence of the curve's length, L_{eff}, "measured" on the resolution scale (or segment length), l_s, as

$$L_{eff} = N_s \, l_s \propto l_s^{1-d_f} \ , \tag{5}$$

where N_s is the number of segments to cover the curve. Consider a Koch curve which starts from a straight line of length L. In each iteration step, each straight segment is augmented into 5 segments of length $L/5$, and "two-sided" roofs are added to the new segments 2 and 4. For example, after 2 iterations this Koch curve looks as follows:

(a) Fill in the table below recording how many segments you need to cover the above curve when decreasing the resolution scale (segment length) l_s by factors of 5.

$l_s[L]$	1				...
N_s	1				...

(b) Use equation (5) and the above table to calculate the fractal dimension of the Koch curve.

6.) *Ising Model* *(20 pts.)*

Consider the 2-D Ising model at vanishing external magnetic field H. In the mean-field approximation, the energy of a single spin, $s_k = \pm 1$ can be written as

$$E_k = -\mu H_{eff} s_k, \qquad (s_k = \pm 1). \tag{6}$$

where the "mean field", $H_{eff} \equiv zJ\langle s\rangle/\mu$, is induced by the z neighboring spins, J is the interaction strength, and $\langle s\rangle$ is the average spin orientation. The latter obeys the following "implicit" equation:

$$\langle s\rangle = \tanh[\frac{\mu H_{eff}}{k_B T}] \ . \tag{7}$$

(a) Sketch the graphical solutions to the mean-field equation for large and small temperature T by plotting the left- and right-hand-side of Eq. (7) versus $\langle s\rangle$.

(b) Expand Eq. (7) for small $\langle s\rangle$ and determine the critical temperature and exponent (*hint*: use $\tanh(x) \simeq x - x^3/3$ for small x). Sketch $\langle s\rangle(T)$ for $T = 0 - 6T_c$.

(c) You now switch to a full numerical simulation by performing Monte-Carlo sweeps through 10×10 spin lattice. When probing a single spin, the energy change to flip it is $E_{flip} = E_{final} - E_{initial}$; describe how the Metropolis algorithm determines whether the spin should be flipped or not.

(d) Using the principle of detailed balance,

$$P_1 W(1 \to 2) = P_2 W(2 \to 1) \tag{8}$$

(P_i: occupation probability of state i, $W(i \to j)$: transition rate from state i to state j), identify the transition rates following from the Metropolis algorithm and show that the resulting occupation probabilities satisfy Boltzmann statistics, i.e.,

$$\frac{P_1}{P_2} = \frac{\mathrm{e}^{-E_1/k_BT}}{\mathrm{e}^{-E_2/k_BT}} \ . \tag{9}$$

7.) *Quantum Mechanics* *(12 pts.)*

Consider the 1-particle Schrödinger equation in 1-dimensinal coordinate space,

$$\left[\frac{1}{2}\frac{d^2}{dx^2} + V(x)\right]\psi(x) = E\psi(x) \ , \tag{10}$$

in units where $m = 1$ and $\hbar = 1$.

(a) Explain why the above equation is more involved than an ordinary 2. order differential equation.

(b) Discretize the Schrödinger equation and solve it for ψ_{n+1} based on the values ψ_{n-1} and ψ_n.

(c) Suppose the potential is a symmetric infinite well, $V(|x| > L) = \infty$ and $V(|x| < L) = 0$. What kind of (space) symmetries does that imply for the wave function, $\psi(x)$, and how can these be exploited to define initial conditions for ψ_0 and ψ_1 when using the "shooting" method of solution?