

FINAL EXAM

PHYS 401 (Spring 2010), 05/11/10

Name:

Signature:

*Duration: 120 minutes*

*Show all your work for full/partial credit!*

1.) *Multiple Choice*

(12 pts.)

For each statement below, circle the correct answer (TRUE or FALSE, no reasoning required).

- (a) The numerical solution to a physics problem must not depend on numerical parameters (e.g., the time-step width) in the code.  
 TRUE                      FALSE
  
- (b) A negative Lyapunov exponent signifies chaotic motion.  
 TRUE                      FALSE
  
- (c) Numerical analysis of planetary motion using Newton's laws cannot test the exponent,  $n = 2$ , in the  $1/r^n$  force law to better than 1%.  
 TRUE                      FALSE
  
- (d) The Feigenbaum number quantifies the rate of period doubling in the transition to chaotic motion.  
 TRUE                      FALSE
  
- (e) If a spanning cluster forms in a percolation transition, that cluster is the only one in the system.  
 TRUE                      FALSE
  
- (f) For the 2-D Ising model at zero external field, the mean-field approximation predicts the critical exponent for magnetization within ca. 20%.  
 TRUE                      FALSE

No.	Points
1	
2	
3	
4	
5	
6	
7	
Sum	

2.) *Realistic Drag Force*

(20 pts.)

Consider a bicycle racer capable of providing a sustained power,  $P_{\text{in}}$ , on a horizontal road on a calm day (no wind). Bicycle+rider have a total mass  $m$ , an effective cross sectional area  $A$  and a drag coefficient  $C$ , resulting in an air-drag force  $F_{\text{drag}} = -B_2 v^2$  where  $B_2 = \frac{1}{2} C \rho_{\text{air}} A$  ( $\rho_{\text{air}}$ : mass density of air).

- (a) For  $C=0$ , use  $F = ma$  and the definition of power,  $P = dE/dt$ , to rewrite the net acceleration as a function of  $P$ . Solve the equation for constant  $P = P_{\text{in}}$  for given initial speed,  $v_0$ , to find the speed as a function of time,  $v(t)$ .
- (b) Include the drag effect in the acceleration and use a first-order difference method to discretize the resulting equation. Express the speed at time step  $i + 1$  in terms of the previous time step  $i$ .
- (c) What is the qualitative difference in the long-time behavior of the speed with and without drag force?
- (d) Why is the solution in part (b) unrealistic for small  $v$  and how can this be improved?

3.) *Newton's Gravitational Law and Planetary Motion*

(15 pts.)

Use Newton's law of gravitation in two dimensions,

$$\vec{F} = -G_N \frac{m_1 m_2}{r^2} \vec{e}_r, \quad (1)$$

to describe the motion of a planet/asteroid (mass  $m_2$ ) around the Sun (mass  $m_1 = M_\odot$ ). Assume the Sun to be fixed in position.

- (a) Write down the two second-order differential equations for the planet's acceleration in  $x$  and  $y$  coordinates using astronomical units ([length] = average Earth-Sun distance; [time] = 1 year; [mass] = solar mass). Specifically, show that  $G_N = 4\pi^2$  in these units.
- (b) Which algorithm would you use to numerically solve these equations and why?
- (c) Kirkwood gaps in asteroid orbits around the Sun occur when the asteroid orbit is in resonance with Jupiter's orbit, i.e., if the two orbit periods are "commensurate",

$$n T_{\text{ast}} = m T_{\text{Jup}} \quad (2)$$

with integer numbers  $n$  and  $m$ . Calculate the orbit radii (in  $AU$ ) for two Kirkwood gaps with  $m = 1$  (Jupiter's orbit radius is  $a_{\text{Jup}} = 5.2 AU$ ). Give an argument why the resonant asteroid orbits can lead to ejection from the solar system.

4.) *Ideal Wave Equation*

(20 pts.)

Consider the 1-D ideal wave equation for the displacement  $y(t, x)$  on a vibrating string as a function of position ( $x$ ) and time ( $t$ ):

$$\frac{d^2 y}{dt^2} = c^2 \frac{d^2 y}{dx^2}, \quad (3)$$

where  $c$  denotes the propagation speed on the string.

- (a) Write down the general form of the solution of the wave equation (3).
- (b) Derive the symmetric finite-difference form of the discretized second-order derivative for an arbitrary smooth function  $f(x)$ .
- (c) Apply your result from part (b) to show that the discretized space-time evolution of  $y$  can be obtained as

$$y_{n+1,i} = r^2(y_{n,i+1} + y_{n,i-1}) + 2(1 - r^2)y_{n,i} - y_{n-1,i}, \quad (4)$$

where the notation  $y_{n,i} \equiv y(t_n, x_i)$  has been used with  $t_n = n\Delta t$  and  $x_i = i\Delta x$ , and  $r = c/(\Delta x/\Delta t)$ . Comment on a suitable choice for  $r$  in the numerical calculations.

- (d) If  $i = 1, \dots, 20$  and  $n = 1, \dots, 30$ , how many discrete initial values  $y_{n,i}$  are needed to solve the time evolution for the entire string?

5.) *Entropy*

(10 pts.)

Consider a diffusion problem of particles on a 2-D grid. For book-keeping purposes partition the system into  $15 \times 15$  equally large cells. Initially, all particles are distributed uniformly over the  $3 \times 3 = 9$  innermost cells in the center of the partition. Imagine you then let the particles randomly diffuse throughout the system.

The 1-particle entropy (or entropy per particle) is defined as

$$S_1 = - \sum_{i=1}^{225} P_i(t) \ln[P_i(t)] \quad (5)$$

in terms of the probability  $P_i(t)$  of finding a particle in one of the 225 cells. Calculate  $S_1$  for  $t = 0$  (initially) and for  $t \rightarrow \infty$  (asymptotically), and use these values to sketch the time evolution of  $S_1(t)$  (no units necessary on the abscissa, i.e., time-axis).

6.) *Koch Curve*

(12 pts.)

The fractal dimension of a curve,  $d_f$ , is defined by the dependence of the curve's length,  $L_{eff}$ , "measured" on the resolution scale (or segment length),  $l_s$ , as

$$L_{eff} = N_s l_s \propto l_s^{1-d_f} , \quad (6)$$

where  $N_s$  is the number of segments to cover the curve. Consider a Koch curve which starts from a straight line of length  $L$ . In each iteration step, each straight segment is divided up into 7 segments of length  $L/7$ , and "squares" are added to the new segments 2, 4 and 6. For example, after 2 iterations this Koch curve looks as follows:

- (a) Fill in the table below recording how many segments you need to cover the above curve when decreasing the resolution scale (segment length)  $l_s$  by factors of 7.

$l_s[L]$	1				...
$N_s$	1				...

- (b) Use equation (6) and the above table to calculate the fractal dimension of the Koch curve.

7.) *Molecular Dynamics and Verlet Method*

(8+3 pts.)

Consider a Molecular Dynamics simulation for a dilute noble gas.

- (a) Using a Taylor expansion, derive the forward-backward symmetric form of the second order derivative (Verlet Method) to numerically solve for the position of each particle in terms of its acceleration and previous coordinates. In particular, show that this method is accurate up to order  $\mathcal{O}(\Delta t^4)$  in the time increment  $\Delta t$ .
- (b) How is the initial speed of the particles related to the equilibrium temperature of the system?