FINAL EXAM

PHYS 401 (Spring 2009), 05/11/09

Name:

Signature:

Duration: 120 minutes

Show all your work for full/partial credit!

1.) *Multiple Choice* *(12 pts.)*

For each statement below, circle the correct answer (TRUE or FALSE, no reasoning required).

(a) When simulating chaotic motion numerically on a computer, the results are not deterministic.
TRUE FALSE

(b) The precession of Mercury's orbit cannot be explained with Newton's laws of motion.
TRUE FALSE

(c) The results of a computational *solution* to a physics problem should be sensitive to numerical input parameters (such as the numerical time-step width, Δt).
TRUE FALSE

(d) All computer-generated random numbers have a finite period.
TRUE FALSE

(e) The concept of entropy cannot be quantified in numerical simulations.
TRUE FALSE

(f) Periodic boundary conditions in numerical simulations (e.g., in the Ising model or in Molecular Dynamics) eliminate finite size effects.
TRUE FALSE

No.	Points
1	
2	
3	
4	
5	
6	
7	
8	
Sum	

2.) *Realistic Drag Force* *(16 pts.)*

Consider a motorbike rider (total mass m) on a horizontal road. Neglect friction but include an airdrag force, $F_d = -B_2 v^2$, depending on the speed, v, of the rider, and a quantity $B_2 = \frac{1}{2} C \rho A$ with C: drag coefficient, ρ: air density, A: effective cross sectional area of rider + bike. The motorbike engine is capable of producing a maximum power P_0.

(a) Use Newton's 2. law of motion and the definition of power, $P = dE/dt$, to rewrite the net acceleration as a function of P.

(b) Use a first-order difference method to discretize the equation derived in (a) and express the speed at time step i in terms of the previous time step $i-1$. Also give the numerical solution for the position, x_i.

(c) What is the terminal speed of the bike, using C=0.5, A=1.2m^2, ρ=1.29m^{-3}, P_0=50kW?

(d) What problem occurs when numerically solving the equation derived in (a) at small v and constant P_0? How can this problem be avoided?

3.) *Chaotic Motion* *(10 pts.)*

Consider a driven, damped, harmonic oscillator.

(a) Explain the concept of the Lyapunov exponent, λ, and how it can be used to characterize whether the oscillator's motion is in a chaotic or regular regime.

(b) Explain what "period doubling" means and how it leads to chaotic motion.

4.) *Logistic Map* *(12 pts.)*

Consider the logistic map defined by the iterative equation

$$x_{n+1} = \mu \, x_n \, (1 - x_n) \tag{1}$$

where n denotes the time step and μ is the "growth" parameter.

(a) Find the first-order fixpoint, x^*, defined by $x = f(x)$ and determine the range of μ values for which the fixpoint is stable. (*hint*: consider small deviations from the fixpoint, $f(x^* + \Delta x) \simeq f(x^*) + f'(x^*)\Delta x$; what condition should the derivative $f'(x^*)$ satisfy to render the fixpoint stable?)

(b) The definition for second-order fixpoints is given by $x = f[f(x)]$. For μ=3.3 it has the solutions $x^* = 0.479, 0.700, 0.824$. Sketch both sides of the equation, $x = f[f(x)]$, graphically vs. x and determine whether the fixpoints are stable or unstable.

5.) *Fourier Analysis* *(10 pts.)*

An engineer samples a time-dependent signal, $y(t)$, N times separated by small time intervals, Δt, giving a discrete signal of real numbers, y_n with $n = 0, \ldots, (N-1)$. He wants to perform a numerical spectral analysis of this signal.

(a) Write down the discretized forms of the Fourier analysis and its inverse,

$$y(t) = \int \frac{d\omega}{2\pi}\, Y(\omega)\; \exp(-i\omega t) \quad , \quad Y(\omega) = \int dt\; y(t)\; \exp(i\omega t)\ , \tag{2}$$

respectively.

(b) The Y_j ($j = 0, \ldots, (N-1)$) are, in general, complex numbers (i.e., $2N$ real numbers), but they are related by $Y_{N-j} = Y_j^*$. What does this imply for the largest frequency (Nyquist frequency) that can be identified from the input signal, y_n? How often is a signal with the Nyquist frequency sampled over its period?

6.) *Fractals* *(12 pts.)*

In the following, the properties of 2- and 1-D fractals will be addressed.

(a) Use the "mass-radius" law to define the fractal dimension of a 2-dimensional object.

(b) The fractal dimension of a curve, d_f, is defined by the dependence of the curve's length, L_{eff}, "measured" on the resolution scale (or segment length), l_s, as $L_{eff} = N_s\, l_s \propto l_s^{1-d_f}$, where N_s is the number of segments to cover the curve. The table below shows the result for $N_s(l_s)$ for a certain Koch curve. Derive and calculate the fractal dimension of this Koch curve.

$l_s[L]$	1	1/5	1/25	1/125	...
N_s	1	9	81	9^3	...

7.) *Ising Model in Mean-Field Approximation* *(16 pts.)*

Consider the 2-D Ising model at vanishing external magnetic field H. In the mean-field approximation, the energy of a single spin s_k can be written as

$$E_k = -\mu H_{eff} s_k, \qquad (s_k = \pm 1). \tag{3}$$

where the "mean field", $H_{eff} \equiv zJ\langle s\rangle/\mu$, is induced by the z neighboring spins, J is the interaction strength, and $\langle s\rangle$ is the average spin orientation. The latter obeys the following "implicit" equation:

$$\langle s\rangle = \tanh[\frac{\mu H_{eff}}{k_B T}] \ . \tag{4}$$

(a) Sketch the graphical solutions to the mean-field equation for large and small temperature T by plotting the left- and right-hand-side of Eq. (4) versus $\langle s\rangle$.

(b) Expand Eq. (4) for small $\langle s\rangle$ and determine the critical temperature and exponent (*hint*: use $\tanh(x) \simeq x - x^3/3$ for small x).

(c) Expand Eq. (4) for small T to find the leading T-dependence of $\langle s\rangle$.

(d) Put the information of parts (b) and (c) together to sketch $\langle s\rangle(T)$ for $T = 0 - 6T_c$.

8.) *Molecular Dynamics and Verlet Method* *(12 pts.)*

Consider a Molecular Dynamics simulation for a dilute noble gas.

(a) Using a Taylor expansion, derive the forward-backward symmetric form of the second order derivative (Verlet Method) to numerically solve for the position of each particle in terms of its acceleration and previous coordinates. In particular, show that this method is accurate up to order $\mathcal{O}(\Delta t^4)$ in the time increment Δt.

(b) Explain how the initial speed of the particles is related to the equilibrium temperature of the system. What subtlety do you need to watch out for in the initialization of the directions of the velocity vectors?