3.) QCD Vacuum + Instantons

3.1. Classical Vacua + Periodicity

3.2. Tunneling Events

3.3. Instanton Vacuum of pure-Glue QCD

3.4. Inclusion of (~massless) Quarks

- Quark $\leftrightarrow$ Instanton Interactions
- Spontaneous Breaking of Chiral Symmetry

References

T. Schäfer + E.V. Shuryak, Rev. Mod. Phys. 70 (1998) 323

D. Diakonov, hep-ph/0212026

3.1 Classical Vacua + Periodicity

"J-V"

def.: $gA_\mu \to A_\mu$, $gG_{\mu\nu} \to G_{\mu\nu}$; Yang-Mills action $S_M = \frac{1}{4g^2}\int d^4x\,(G^a_{\mu\nu})^2 = \frac{1}{2g^2}\int d^4x\,(\vec{E}^2 - \vec{B}^2)$

(temporal gauge $A_0 \equiv 0$: $\vec{E}_a = \dot{\vec{A}}_a$, $\vec{B}_a = \vec{\partial}\times\vec{A}_a + \frac{\varepsilon^{abc}}{2}\vec{A}_b\times\vec{A}_c$)

$$S_E \equiv -i S_M(t=-i\tau) = \frac{1}{2g^2}\int d^4x\,(\vec{E}^2 + \vec{B}^2)$$

objective: determine the "true" vacuum state !

Start from 2 basic observations:

(1) Distinct classical vacuum states

$\to$ $G^a_{\mu\nu} G^{\mu\nu}_a = 0$, but underlying gauge fields can differ:

$A_\mu \to U^+ A_\mu U + \underbrace{i\, U^+ \partial_\mu U}$, $U \in SU(3)$, time-independent

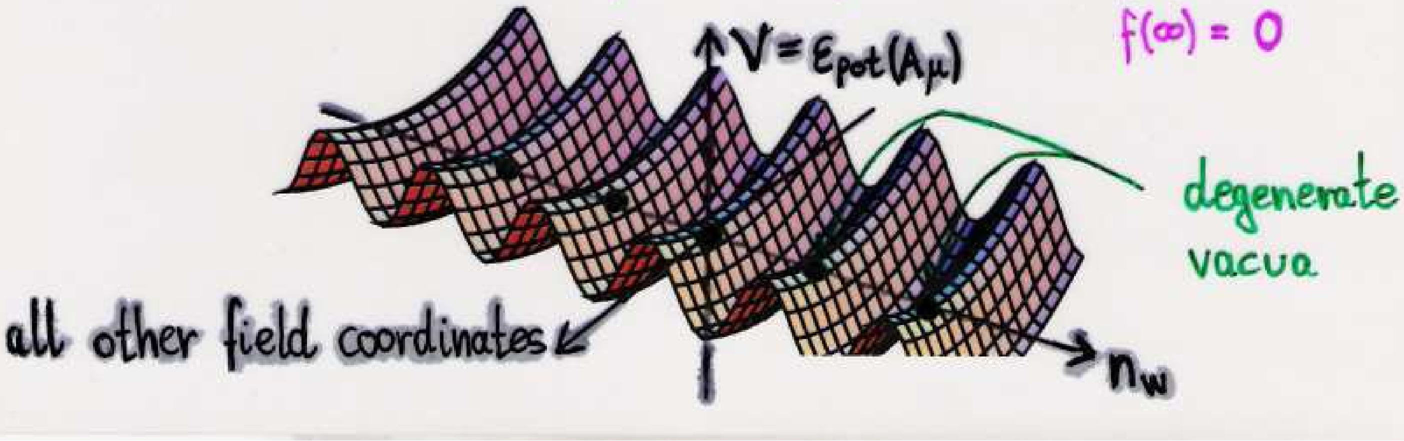
"pure gauge" can be characterized by "winding number":

$$\boxed{n_w \equiv \frac{1}{24\pi^2}\,\varepsilon^{ijk}\int d^3x\,\mathrm{tr}\left[(U^+\partial_i U)(U^+\partial_j U)(U^+\partial_k U)\right]} \in \mathbb{Z}$$

$$= (1/16\pi^2)\,\varepsilon^{ijk}\int d^3x\,\left(A^a_i\,\partial_j A^a_k + \tfrac{1}{3} f^{abc} A^a_i A^b_j A^c_k\right)$$

e.g. 1-D : $U = \exp(in\alpha)$ $\Rightarrow$ $n_w = (1/2\pi)\int_0^{2\pi} U^+\partial_\alpha U$

3-D QCD: $U = \exp[i f(r)\,\vec{\tau}\cdot\vec{r}/r]$ with $f(0) = n\pi$, $f(\infty) = 0$



(2) Properties of "Dual" Field Strength

$$\boxed{\tilde{G}^a_{\mu\nu} \equiv \frac{1}{2}\,\varepsilon_{\mu\nu\sigma\tau}\,G^a_{\sigma\tau}}$$

can be used to construct total divergence:

$$\frac{1}{32\pi^2} G^a_{\mu\nu}\tilde{G}^a_{\mu\nu} \equiv \partial_\mu K_\mu \quad \text{with} \quad K_\mu = \frac{1}{16\pi^2}\varepsilon_{\mu\alpha\beta\gamma}\left(A^a_\alpha \partial_\beta A^a_\gamma + \frac{1}{3} f^{abc} A^a_\alpha A^b_\beta A^c_\gamma\right)$$

(note: $\int K_0\, d^3x = n_w$!)

$\rightarrow$ action for <u>any</u> $G^a_{\mu\nu}$:

$$S = \frac{1}{4g^2}\int d^4x \left[\pm G^a_{\mu\nu}\tilde{G}^a_{\mu\nu} + \frac{1}{2}\left(G^a_{\mu\nu} \mp \tilde{G}^a_{\mu\nu}\right)^2\right]$$

$$= \frac{8\pi^2}{g^2}\int d^4x\,(\partial_\mu K_\mu) + \underbrace{\frac{1}{8g^2}\int d^4x\,(G-\tilde{G})^2}_{\geq 0}$$

(3) Consider field configuration $G(\tau = \pm\infty) \,\hat{=}\,$ pure gauge (classical vacuum)

i.e. $G(\tau) \,\hat{=}\,$ vac. $\rightarrow$ vac. transition:

$$\boxed{S - \frac{1}{8g^2}\int (G-\tilde{G})^2} = \frac{8\pi^2}{g^2}\int d^4x\left(\partial_0 K_0 + \vec{\partial}\cdot\vec{K}\right)$$

$$= \frac{8\pi^2}{g^2}\left[\int d^3x\, K_0\right]_{\tau=-\infty}^{\tau=+\infty} + \frac{8\pi^2}{g^2}\int dt \underbrace{\int_{S_2} d\vec{S}\cdot\vec{K}}_{0}$$

$A(r \to \infty) = 0$

$$= \frac{8\pi^2}{g^2}\left[n_w(+\infty) - n_w(-\infty)\right] = \boxed{\frac{8\pi^2}{g^2}|Q|}$$

"topological charge"

<u>implies</u>:

(i) minimal (eucl.) action for selfdual fields $(G = \tilde{G})$

(ii) $S = (8\pi^2/g^2)\,|Q|$ for vacuum $\rightarrow$ vacuum

(iii) selfdual fields satisfy YM-equation of motion (since $D_\mu \tilde{G}_{\mu\nu} \equiv 0$)

$\Rightarrow$ **Tunneling between top. different vacua!** (energy conserved, no ext. current)

3.2 (Re-) Construction of Tunneling Events

ansatz: 4-D mapping (includes time-dependence)

$$U \stackrel{!}{=} \frac{i x_\mu \tau_\mu^+}{\sqrt{x^2}}$$

definitions $\tau_\mu^\pm \equiv (\vec{\tau}, \pm i)$

$$\Rightarrow \tau_\mu^\pm \tau_\nu^\mp = \delta_{\mu\nu} + i \begin{Bmatrix} \eta_{\mu\nu}^a \\ \bar{\eta}_{\mu\nu}^a \end{Bmatrix} \tau^a$$

't Hooft symbol
$$\eta_{\mu\nu}^a = \begin{cases} \varepsilon_{a\mu\nu} & \mu,\nu \in \{1,2,3\} \\ \pm\delta_{a\mu} & \nu = 4 \\ \mp\delta_{a\nu} & \mu = 4 \end{cases}$$

$\rightarrow$

$$A_\mu = i U^+ \partial_\mu U = 2 \eta_{\mu\nu}^a \frac{x_\nu}{x^2} \left(\frac{\lambda^a}{2}\right)$$

still pure gauge with $n_w = 1$

$\rightarrow$

render it physical: $\boxed{A_\mu^a \stackrel{!}{=} 2 \eta_{\mu\nu}^a f(x^2) \frac{x_\nu}{x^2}}$ $\quad f(x^2 \to \infty) \stackrel{!}{=} 1$

$\rightarrow$ evaluate field strength:

$$G_{\mu\nu}^a = -4 \left[\eta_{\mu\nu}^a \frac{f(1-f)}{x^2} + (x_\mu \eta_{\nu\sigma}^a - x_\nu \eta_{\mu\sigma}^a) x_\sigma \frac{f(1-f) - x^2 f'}{x^4} \right]$$

$$\tilde{G}_{\mu\nu}^a = -4 \left[\eta_{\mu\nu}^a f' + \text{———} \;"\; \text{———} \right]$$

selfduality $\Rightarrow$ $x^2 f' \stackrel{!}{=} f(1-f)$ $\Rightarrow$ $f(x^2) = \frac{x^2}{(x^2+\rho^2)}$

ρ: "radius" (integration constant)

$$\boxed{\begin{aligned} \Rightarrow G_{\mu\nu}^a &= (-4)\, \eta_{\mu\nu}^a \frac{\rho^2}{(x^2+\rho^2)^2} \\ (G_{\mu\nu}^a)^2 &= \frac{192 \rho^4}{(x^2+\rho^2)^4} \end{aligned}}$$

Instanton field configuration!

$(\eta^2 = 12)$

check action: $S = \frac{1}{4g^2} \int d^4x \, \frac{192 \rho^4}{(x^2+\rho^2)^4} = \frac{8\pi^2}{g^2} = \frac{2\pi}{\alpha_s}$ $\quad (\simeq 10-15)$

tunneling probability $P \propto e^{-S}$ $\quad$ small ?!

pre-exponent ?

3.3 Pure-Glue QCD : Instanton Vacuum

partition function $\mathcal{Z} = \int \mathcal{D}A_\mu \, e^{-S_E(A_\mu)}$

↓

expand around instanton action : $S_E = S_{inst} + \delta S$ ← quantum

i.e. $A_\mu = A_\mu^{inst} + \delta a_\mu$ ← fluctuations

(i) $\delta S = 0$ in direction of instanton "coordinates" : $\left.\begin{array}{l} 1 \text{ size } \rho, \ 4 \text{ position } z_\mu \\ 7(3) \text{ color angles in } SU(3)(2) \end{array}\right\} 4N_c$

→ transform to collective coordinates $\Omega_I = \{\rho_I, z_I, U_I\}$ ⇒ Jacobian $(\sqrt{S_{inst}})^{4N_c}$

(ii) all other field coordinates in Gaussian approx. ⇒ factor C_{N_c}

⇒ tunneling probability for 1 instanton :

$$\boxed{dn_I = d\rho \, d^4z \, \underbrace{C_{N_c} \left(\frac{8\pi^2}{g^2}\right)^{2N_c} e^{-8\pi^2/g^2(\rho)} \frac{1}{\rho^5}}_{\equiv n(\rho)} e^{-S_{int}}}$$

QCD running coupling $\frac{8\pi^2}{g^2(\rho)} = \frac{2\pi}{\alpha_s(\rho)} = b \log\left(\frac{1}{\Lambda\rho}\right)$ → $(\Lambda\rho)^b$ $\quad b = \frac{11}{3}N_c - \frac{2}{3}N_f$

→ repulsive (gluonic) I-I interaction to stabilize sizes: $S_{int} \propto \frac{N}{V_4} \bar{\rho}^2 \rho^2$

Anti-/Instanton Vacuum

≙ Grand Canonical Ensemble of "Pseudoparticles"

$$\mathcal{Z}_{inst} = \sum_{N_+, N_-} \frac{1}{N_+! \, N_-!} \prod_{I, \bar{I}} d\Omega_I \, n(\rho_I) \, e^{-S_{int}}$$

(in practice: minimize canonical $\mathcal{Z}_{inst}(N_+ = N_-)$ w.r.t. $\frac{N}{V_4} = (N_+ + N_-)/V_4$)

By how much does tunneling reduce the ground state (vacuum) energy ?

Free Energy in the Pure-Glue Instanton Vacuum

$$\Omega = -\frac{1}{V_4} \log \mathcal{Z}$$

Mean-Field Approximation:

→ individual anti-/instantons with average repulsion

$$\mathcal{Z}_{inst}^{canonical} \simeq \frac{1}{N_+!\,N_-!} (V_4 z_+)^{N_+} (V_4 z_-)^{N_-}$$

1-I activity: $z_\pm = \int d\rho \, \underbrace{n_\pm(\rho)\, e^{-\kappa \rho^2 \bar{\rho}^2 \frac{N}{V_4}}}_{\equiv \mu_\pm(\rho)}$

average size: $\bar{\rho}^2 = \frac{1}{z_\pm} \int d\rho\, \rho^2 \mu_\pm(\rho)$

selfconsistent: $\bar{\rho}^2 = \left(\frac{1}{\kappa}\frac{V_4}{N}\nu\right)^{1/2}$, $\nu = \frac{1}{2}(b-4)$

$$\Rightarrow \mu_\pm(\rho) = n_\pm(\rho)\, e^{-\nu \rho^2/\bar{\rho}^2}$$

$$\langle S_{int} \rangle = \nu = 3.5$$

$$z_+ = B(N_c, \Lambda_{QCD})\, n_+^{-\nu/2} \equiv z_- \equiv \frac{z}{2}$$

$$N_+ = V_4 n_+ = N_- = N/2 = V_4 \frac{n}{2}$$

evaluate free energy (use $\log(N_\pm!) \simeq N_\pm (\log N_\pm - 1)$):

$$\boxed{\Omega_{inst}^{can.}} = -\frac{N}{V_4}\left[-\log\left(\frac{N}{2}\right) + 1 + \log\left(V_4 \frac{z}{2}\right)\right]$$

$$= \boxed{n\left[\log\left(\frac{n}{z(n)}\right) - 1\right]}$$

grand canonical: $\frac{\partial \Omega}{\partial n} \stackrel{!}{=} 0 \quad \Rightarrow \quad n_{min} = (2B\, e^{-\nu/2})^{2/(\nu+2)}$

$$\Rightarrow \boxed{\Omega_{inst}^{min} = -n_{min}\left(1 + \frac{\nu}{2}\right) = -\frac{b}{4} n_{inst}} \simeq -(0.5\text{-}0.8)\ \mathrm{GeV/fm^3}$$

lattice QCD: $n_{inst} \simeq (1\text{-}1.4)\,\mathrm{fm^{-4}}$ ⟷ $\Lambda_{QCD} \simeq 250\ \mathrm{MeV}$

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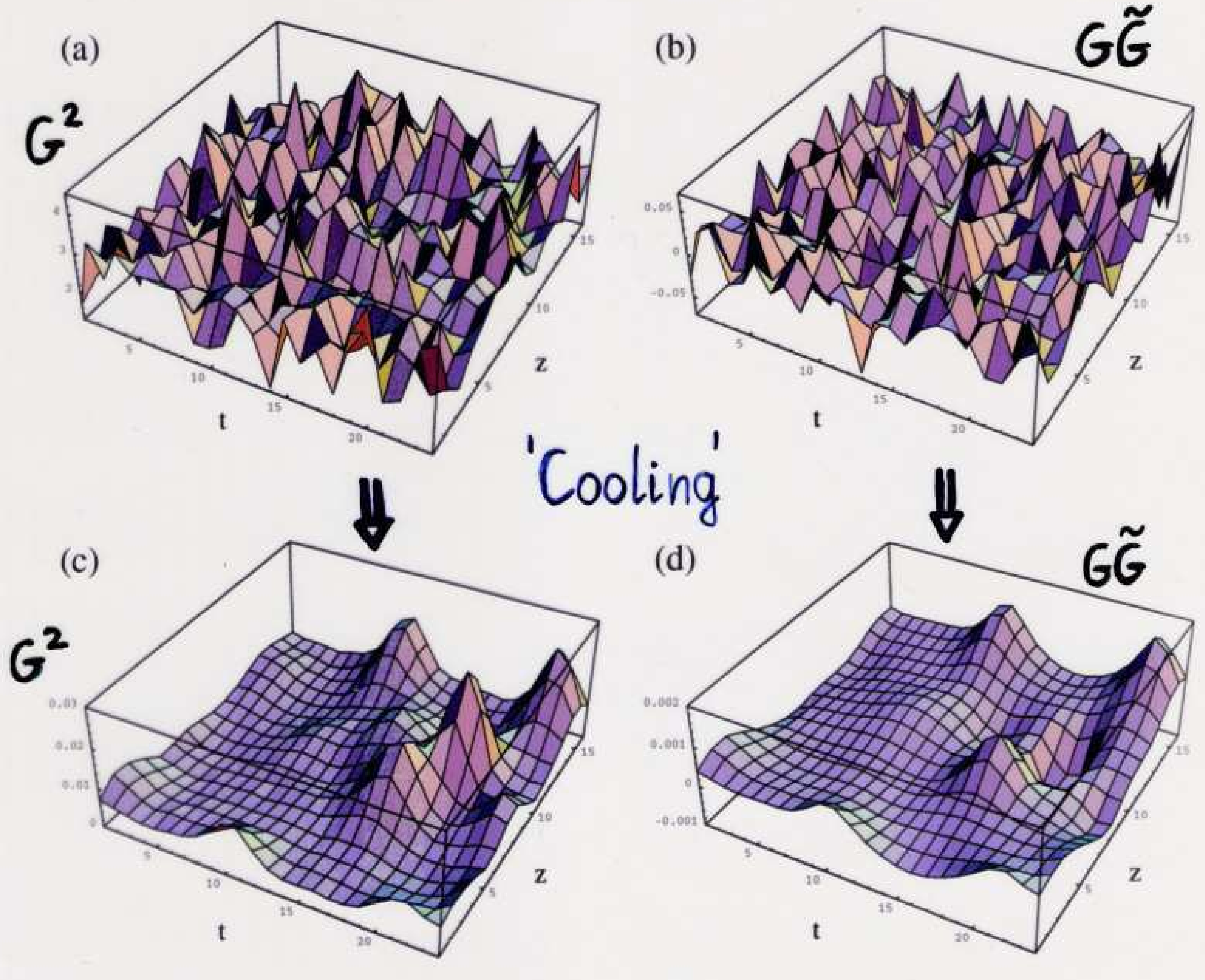
Instantons on the Lattice ($T = \mu_q = 0$)



Field Strength   Topological Charge

M.C. Chu, J.M. Grandy, S. Huang + J.W. Negele,
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