Thermal Field Theory and Instantons

...in QCD

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1.) Introduction: QCD and its Phase Diagram

2.) Elements of Finite-T Field Theory
   - Partition Function + Free Fields
   - Interactions: Mean-Field Approximation

3.) Instantons + QCD Vacuum
   - Single-Instanton Solution + Interpretation
   - Instanton Interactions + Ensemble
   - Light Quarks + Chiral Symmetry Breaking

4.) Finite-T Chiral Phase Transition

5.) High-Density QCD
   - Color Superconductivity
   - Crystalline Phases
   - Instanton Ensemble at finite μ_q

6.) Summary

1.) Introduction I :
The Origin of Mass in Matter

Elementary Building Blocks :

Fermions:

- quarks $\left\{ \begin{matrix} u & c & t \\ d & s & b \end{matrix} \right.$
- leptons $\left\{ \begin{matrix} e & \mu & \tau \\ \nu_e & \nu_\mu & \nu_\tau \end{matrix} \right.$

$\left. \begin{matrix} g \\ \gamma \\ W \\ Z \end{matrix} \right\}$ Gauge Bosons

masses from electroweak gauge-symmetry breaking:

$m_f \propto \langle 0|\phi|0\rangle \neq 0$ (Higgs particle(s))

Stable Matter : only e^-, u, d $(m \simeq \frac{1}{2} - 10\,\mathrm{MeV})$

$\Rightarrow$ 2 Questions :

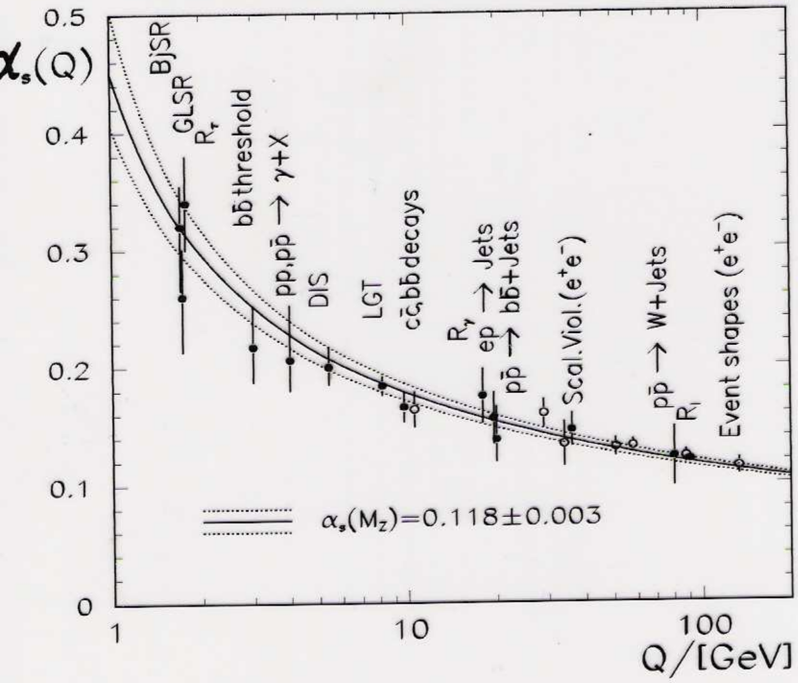
1. single quarks not observed
   $\hat{=}$ 'Confinement': only (qqq) baryons, $(q\bar{q})$ mesons
2. 
   proton $\hat{=}$ (uud)
   neutron $\hat{=}$ (ddu)
   $M_{p,n} \gg m_{u,d}$
   $1\,\mathrm{GeV} \gg 10\,\mathrm{MeV}$
   $\hat{=}$ 'Chiral Symmetry Breaking'

} answers in the theory of Strong Interactions

Intro-II: Quantum ChromoDynamics (QCD)

$$\boxed{\mathcal{L}_{QCD} = \bar{q}\,(i\not\partial - ig\not A - \hat{m}_q)\,q - \frac{1}{4}G_{\mu\nu}G^{\mu\nu}}$$

↳ well-tested at high energies ($Q^2 > 1\,\mathrm{GeV}^2$):



perturbation theory !
(pQCD)

$\alpha_s = \frac{g^2}{4\pi}$ small

degrees of freedom
$\hat{=}$ elementary q, g

$Q^2 \lesssim 1\,\mathrm{GeV}^2$: transition to 'strong' QCD

⇒ 2 Main Phenomena:

(1) effective degrees of freedom $\hat{=}$ hadrons

Confinement

(2) 'constituent' quarks appear to be massive
$m_q^* \simeq (300-400)\,\mathrm{MeV} \simeq \frac{1}{3}M_N$

Spontaneous Breaking of Chiral Symmetry

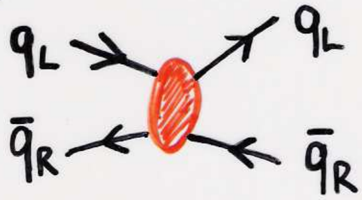
Intro-III: Chiral Symmetry in QCD

$$\boxed{\mathcal{L}_{QCD}^{light} = (\bar{u}_L, \bar{d}_L)\, \not{D} \begin{pmatrix} u_L \\ d_L \end{pmatrix} + (\bar{u}_R, \bar{d}_R)\, \not{D} \begin{pmatrix} u_R \\ d_R \end{pmatrix} + \mathcal{O}(m_{u,d})}$$

$\hat{=}$ spin-flavor ('chiral') symmetry: $SU(2)_L \otimes SU(2)_R$

... and its Spontaneous Breaking

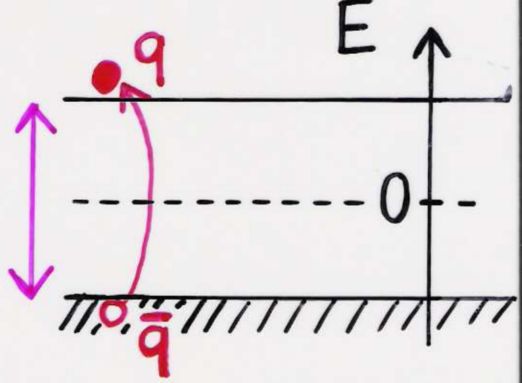
strong attraction in scalar $q\bar{q}$-channel



$\Downarrow$

Bose Condensate $\langle\bar{q}q\rangle \neq 0$ $= \langle \bar{q}_L q_R + \bar{q}_R q_L \rangle$ fills **QCD** vacuum !

Profound Consequences

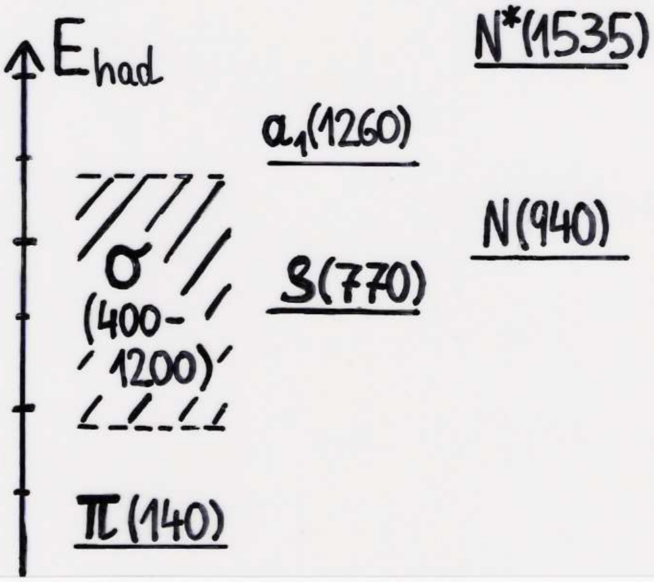


- energy gap $E_{\bar{q}q} \simeq 2 m_q^* \propto \langle \bar{q} q \rangle$
- massless Goldstone Bosons $\pi^{\pm,0}$
- SU(2) chiral multiplets split

$J^P = 0^\pm$ $\quad \pi \leftrightarrow \sigma$

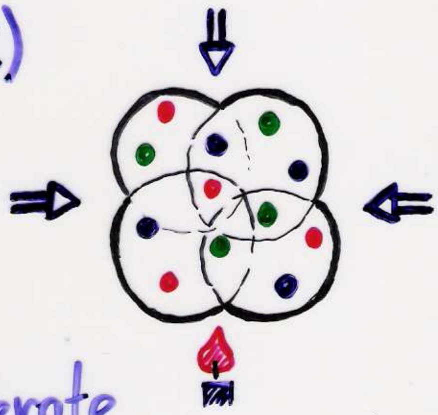
$1^\pm$ $\quad \rho \leftrightarrow a_1$ $\qquad \Delta E_h \simeq 0.5\,\text{GeV}$

$\frac{1}{2}^\pm$ $\quad N \leftrightarrow N^*$



3.) Probing the QCD Phase Diagram

excite vacuum (heat, compress nuclei)

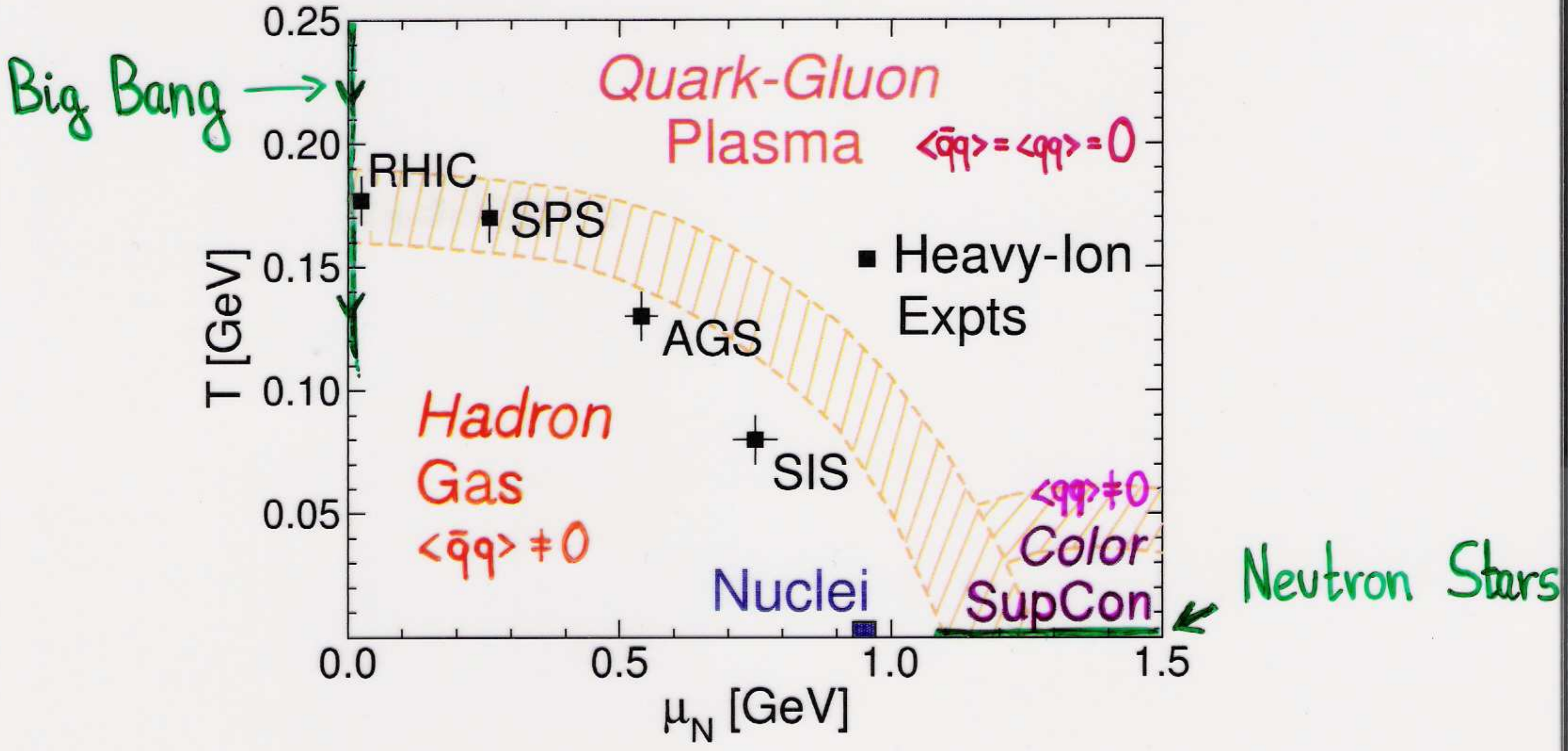


$\Downarrow$ finite T, ρ_B

- quarks "percolate" + liberated
- $\langle \bar{q}q \rangle$ 'melts', chiral multiplets degenerate

$\Rightarrow$ medium modifications of hadrons $\hat{=}$ precursors of chiral symmetry restoration

Phase Structure of Strong Interaction Matter



$\Rightarrow$ heavy-ion collisions ('Little Bang' $\hat{=}$ 10^{-6} s after 'Big Bang')

$\Rightarrow$ neutron stars $\rightarrow$ high-density, low-temperature frontier

2.) Elements of Finite-Temperature Field Theory

2.1. Partition Function

- Analogy to QFT Amplitudes (Vacuum)
- Imaginary Time + Periodicity
- Propagators

2.2. Thermodynamic Potential for Free Fields

- Bosons
- Fermions

2.3. Interactions

- Mean-Field Approximation
  + Ground-State Condensates

References:

J. Kapusta, "Finite-Temperature Field Theory"
M. Le Bellac, "Thermal Field Theory"

2.1 Finite-Temperature Partition Function

(1) Transition Amplitude in Quantum Field Theory (Vacuum)

$$A_{a\to b}(t) \equiv \langle \phi_b(t) | \phi_a(0)\rangle = \langle \phi_b(t) | \underbrace{e^{-i\hat{H}t}}_{} | \phi_a(0)\rangle$$

time evolution operator $\left(\hat{H}|\phi\rangle = i\frac{\partial}{\partial t}|\phi\rangle\right)$

$\hat{H} \equiv \int d^3x\, \mathcal{H}(\hat{\phi},\hat{\pi})$ "coordinates" $\hat{\phi}(\vec{x},0)|\phi\rangle = \phi(\vec{x})|\phi\rangle$

"conjugate momenta" $\hat{\pi}(\vec{x},0)|\pi\rangle = \pi(\vec{x})|\pi\rangle$

overlap: $\langle\phi|\pi\rangle = \exp\left[i\int d^3x\, \pi(x)\phi(x)\right]$ $\left(\langle x|p\rangle = e^{ip\cdot x}\right)$

eliminate operators:

insert complete sets in intervals Δt : $\cdots\int d\pi_i\, d\phi_i\, |\pi_i\rangle \underbrace{\langle\pi_i| e^{-i\hat{H}\Delta t}|\phi_i\rangle}_{(1-iH_i\Delta t)\langle\pi_i|\phi_i\rangle} \langle\phi_i|\cdots$

$$\Rightarrow \boxed{A_{a\to b}} = \int \mathcal{D}\pi \int_{\phi_a}^{\phi_b} \mathcal{D}\phi\, \exp\left[i\int d^4x\left(\pi\dot{\phi} - \mathcal{H}(\pi,\phi)\right)\right]$$

$$\boxed{= \text{const} \int_{\phi_a}^{\phi_b} \mathcal{D}\phi\; e^{iS(\phi,\dot{\phi})}}$$

Action $S = \int_0^t dt \int d^3x\, \mathcal{L}(\phi,\dot{\phi})$

path integral!

(2) Partition Function in Statistical Mechanics

equilibrium (temperature $T \equiv 1/\beta$) $\leftrightarrow$ steady state

$$\mathcal{Z} \equiv \text{Tr}\exp\left[-\underbrace{\beta(\hat{H}-\mu\hat{N})}_{\hat{=}\int_0^\beta d\tau\int d^3x(\hat{\mathcal{H}}-\mu\hat{\mathcal{N}})}\right] = \int d\phi_a \langle\phi_a| e^{-\beta(\hat{H}-\mu\hat{N})}|\phi_a\rangle$$

no "i"

"imaginary time" $it \hat{=} \tau$

$$= \int_{\phi(\vec{x},0)\overset{!}{=}\pm\phi(\vec{x},\beta)} \mathcal{D}\phi\, \exp\left[+\int_0^\beta d\tau\int d^3x\left(\mathcal{L}+\mu\mathcal{N}\right)\right] \equiv \int_{\text{periodic}} \mathcal{D}\phi\; e^{-S_E}$$

euclidean action

definition of euclidean space-time

$$\left.\begin{aligned} \tau \equiv x_4 &\equiv i x_0 \\ k_4 &\equiv -i k_0 \end{aligned}\right\} \Rightarrow \begin{aligned} d^4x &= -i\, d^4x_E \\ d^4k &= +i\, d^4k_E = i\, d^3k\, dk_4 \quad , \; k^2 = -k_4^2 - \vec{k}^2 = -k_E^2 \end{aligned}$$

$iS = -S_E$ euclidean ("imaginary time") action

e.g. $iS = i\int d^4x \, \frac{1}{2}\left(\partial_\mu\phi\,\partial^\mu\phi - m^2\phi^2\right)$ (free bosons)

$$-S_E = +\int d^4x_E \, \frac{1}{2}\left(-(\partial_\mu^E\phi)^2 - m^2\phi^2\right) = -\int d^4x_E \, \frac{1}{2}\left((\partial_\mu^E\phi)^2 + m^2\phi^2\right) \leq 0$$

Field (anti-) periodicity $\phi(\vec{x},0) = \pm\phi(\vec{x},\beta)$ implies

$$\phi(\vec{x},\tau) = \left(\frac{1}{V}\right)^{\frac{1}{2}} \sum_{n=-\infty}^{+\infty} \sum_{\vec{p}} e^{i(\vec{p}\cdot\vec{x} - \omega_n\tau)}\,\phi_n(\vec{p}) \quad \text{with } \omega_n = \begin{cases} 2n\frac{\pi}{\beta} & \text{for "+"} \\ (2n+1)\frac{\pi}{\beta} & \text{for "-"} \end{cases}$$

"Matsubara frequencies"

Propagators (Green's Functions)

→ defined as thermal expectation value

$$G(\vec{x},\vec{y};\tau,0) \equiv \mathcal{Z}^{-1}\,\mathrm{Tr}\left(\hat{\varrho}\; T_\tau\left[\hat{\phi}(\vec{x},\tau)\,\hat{\phi}(\vec{y},0)\right]\right) \quad , \quad \tau \in [0,\beta]$$

density matrix: $\hat{\varrho} \equiv e^{-\beta\hat{K}} \equiv e^{-\beta(\hat{H} - \mu\hat{N})}$ $\quad (\mathcal{Z} = \mathrm{Tr}\,\hat{\varrho})$

time-ordering operator:

bosons $\quad T_\tau\left[\hat{\phi}(\tau_1)\,\hat{\phi}(\tau_2)\right] = \hat{\phi}(\tau_1)\,\hat{\phi}(\tau_2)\,\theta(\tau_1-\tau_2) + \hat{\phi}(\tau_2)\,\hat{\phi}(\tau_1)\,\theta(\tau_2-\tau_1)$

fermions $\quad T_\tau\left[\hat{\Psi}(\tau_1)\,\hat{\Psi}(\tau_2)\right] = \quad$ " $\quad - \quad$ "

Field Anti-/Periodicity + Greens Functions

$$G^B(\vec{x},\vec{y};\tau,\underline{\underline{\beta}}) = \mathcal{Z}^{-1}\,\mathrm{Tr}\,\big(e^{-\beta\hat{K}}\,\overbrace{\hat{\phi}(\vec{y},\beta)}^{e^{+\beta\hat{K}}\,\hat{\phi}(\vec{y},0)\,e^{-\beta\hat{K}}}\,\hat{\phi}(\vec{x},\tau)\big) \qquad (\tau<\beta)$$

$$= \mathcal{Z}^{-1}\,\mathrm{Tr}\,\big(\hat{\phi}(\vec{y},0)\,e^{-\beta\hat{K}}\,\hat{\phi}(\vec{x},\tau)\big) \quad \boxed{= +\,G^B(\vec{x},\vec{y};\tau,\underline{\underline{0}})}$$

likewise: $G^B(\vec{x},\vec{y};\tau-\beta,0) = G^B(\vec{x},\vec{y};\tau,0) \iff \boxed{\begin{array}{l}\hat{\phi}(\vec{y},\beta) = +\hat{\phi}(\vec{y},0)\\ \Rightarrow \omega_n^B = 2n\pi T\end{array}}$

Bosons

but: $G^F(\vec{x},\vec{y};\tau,\beta) = -\,G^F(\vec{x},\vec{y};\tau,0) \iff \boxed{\begin{array}{l}\hat{\Psi}(\vec{y},\beta) = -\hat{\Psi}(\vec{y},0)\\ \Rightarrow \omega_n^F = (2n+1)\pi T\end{array}}$

Fermions

2.2 Thermodynamic Potential for Non-Interacting Particles

Free Energy Density $\boxed{\frac{F}{V} \equiv \Omega = \frac{(-1)}{V_4} \log \mathcal{Z} = \frac{-T}{V} \log \mathcal{Z}}$

Pressure $P = -\frac{\partial F}{\partial V} = -\frac{\partial}{\partial V}(\Omega V) = -\Omega$

entropy density $s = -\frac{\partial \Omega}{\partial T}$, energy density $\varepsilon = \frac{\partial(\beta\Omega)}{\partial \beta} = -P + Ts$

(1) Free Scalar (Boson) Field

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$$

$$\Rightarrow S_E = \frac{1}{2}\int_0^\beta d\tau \int d^3x \left[\left(\frac{\partial\phi}{\partial\tau}\right)^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2 \right]$$

$$= \frac{1}{2}\int_0^\beta d\tau \int d^3x \, \phi \underbrace{\left[-\frac{\partial^2}{\partial\tau^2} - \vec{\nabla}^2 + m^2\right]}_{\equiv \hat{D}_{KG} \text{ : Klein-Gordon operator (euclidean)}} \phi$$

expand: $\phi(x,\tau) = \left(\frac{\beta}{V}\right)^{\frac{1}{2}} \sum_n \sum_{\vec{p}} e^{i(\vec{p}\cdot\vec{x} - \omega_n \tau)} \phi_n(\vec{p}) \quad \in \mathbb{R}$

$\Rightarrow$ phases: $\phi_{-n}(-\vec{p}) = \phi_n^*(\vec{p})$

$$\boxed{S_E = \frac{1}{2}\beta^2 \sum_n \sum_{\vec{p}} (\omega_n^2 + \omega_p^2)\, A_n(\vec{p})^2}$$

$A_n(\vec{p}) = |\phi_n(\vec{p})|$

$\omega_p^2 = \vec{p}^2 + m^2$

$$\Rightarrow \mathcal{Z}_B = \int \mathcal{D}\phi \, e^{-S_E} = \prod_n \prod_{\vec{p}} \int_{-\infty}^{+\infty} dA_n(\vec{p}) \exp\left[-\tfrac{1}{2}\beta^2(\omega_n^2 + \omega_p^2) A_n(\vec{p})^2\right]$$

$$= \text{const} \prod_n \prod_{\vec{p}} \left[\beta^2(\omega_n^2 + \omega_p^2)\right]^{-1/2} \qquad \text{using } \int dA\, e^{-xA^2} = \sqrt{\frac{\pi}{x}}$$

$$\Rightarrow \log \mathcal{Z}_B = -\frac{1}{2}\sum_n \sum_{\vec{p}} \log\left[\beta^2(\omega_n^2 + \omega_p^2)\right] = -\frac{1}{2}\sum_n \sum_{\vec{p}} \int \frac{d\omega_p^2}{\omega_n^2 + \omega_p^2}$$

$$= -\frac{1}{2}\sum_{\vec{p}} \int d\omega_p^2 \frac{\beta}{\omega_p}\left(\frac{1}{2} + f^B(\omega_p)\right) \qquad = \frac{\beta}{2\omega_p}\left(1 + 2f^B(\omega_p)\right)$$

$$\Omega_B = -\frac{T}{V}\log Z_B = \boxed{T\int\frac{d^3p}{(2\pi)^3}\left[\frac{1}{2}\beta\omega_p + \log(1-e^{-\beta\omega_p})\right]}$$

zero-point ↑ ↑ thermal

massless limit

$$-P_0 - \frac{\pi^2}{90}T^4$$

Thermodynamics

$$s = -\frac{\partial\Omega}{\partial T} = \int\frac{d^3p}{(2\pi)^3}\left[(1+f^B)\log(1+f^B) - f^B\log f^B\right] \qquad \frac{4\pi^2}{90}T^3$$

$$\varepsilon = \frac{\partial(\beta\Omega)}{\partial\beta} = \int\frac{d^3p}{(2\pi)^3}\,\omega_p\left[\frac{1}{2}+f^B\right] \qquad \varepsilon_0 + \frac{\pi^2}{30}T^4$$

$$f^B = (e^{\beta\omega_p}-1)^{-1}$$

$$\boxed{\Omega = -P = \varepsilon - Ts}$$

more compact notation

$$\boxed{Z_B} = \int\mathcal{D}\phi\, e^{-S_E} = \int\mathcal{D}\phi\, e^{-\frac{1}{2}(\phi,\hat{D}_{KG}\phi)} \leftarrow \text{"scalar product"}$$

$$= \int d\phi_1\cdots d\phi_n\, e^{-1/2\,\phi_i D_{ij}\phi_j}$$

choose eigenstates

$$\hat{D}\phi_i = \lambda_i\phi_i$$

$$= \int d\phi_1\, e^{-\frac{1}{2}\lambda_1\phi_1^2}\cdots\int d\phi_n\, e^{-\frac{1}{2}\lambda_n\phi_n^2} = \frac{\text{const}}{\prod_{i=1}^{n}\sqrt{\lambda_i}} = \boxed{\frac{\text{const}}{(\det\hat{D}_{KG})^{1/2}}}$$

(2) Free Spin-1/2 Fields (Fermions)

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi \equiv \bar{\Psi}\hat{D}\Psi \qquad \text{Dirac operator}$$

conjugate momentum $\Pi = \frac{\partial\mathcal{L}}{\partial\dot{\Psi}} = i\Psi^+ \equiv \Psi_E^+$

also: $-i\gamma_k \equiv \gamma_k^E$, $it \equiv \tau$

$$\Rightarrow \mathcal{L} = (-i)\bar{\Psi}_E(-\partial\!\!\!/_E - m)\Psi_E = -\bar{\Psi}_E(-i\partial\!\!\!/_E - im)\Psi_E \equiv -\mathcal{L}_E$$

$$\Rightarrow Z_F = \int\mathcal{D}\Psi_E^+\mathcal{D}\Psi_E\, e^{-S_E} = \int\mathcal{D}\Psi_E^+\mathcal{D}\Psi_E\,\exp\left[-\int_0^\beta d\tau\int d^3x\,(\mathcal{L}_E + i\mu\Psi_E^+\Psi_E)\right]$$

anti-commutation relations for Grassmann fields

$$\{\hat{\Psi}_\alpha(\vec{x},t), \hat{\Psi}^+_\beta(\vec{y},t)\} = \delta(\vec{x}-\vec{y})\,\delta_{\alpha\beta} \quad ; \quad \{\hat{\Psi}_\alpha, \hat{\Psi}_\beta\} = \{\hat{\Psi}^+_\alpha, \hat{\Psi}^+_\beta\} = 0$$

consequence for functional integration (single state):

$$\int d\Psi_1\, \Psi_1 = \int d\Psi_1 (\Psi_1 + \text{const}) = 1 \quad , \quad \int d\Psi_1\, \Psi_1^n = 0 \;,\; n \geq 2$$ ("Pauli Principle")

Example: 2+2 states

$$\boxed{\int \mathcal{D}\Psi^+ \mathcal{D}\Psi\, e^{\Psi^+ \hat{D} \Psi}} = \int d\Psi_1^+ d\Psi_1\, d\Psi_2^+ d\Psi_2\, e^{\Psi_i^+ D_{ij} \Psi_j}$$

non-vanishing

$$= \int d\Psi_1^+ d\Psi_1\, d\Psi_2^+ d\Psi_2 \left(1 + [\Psi_1^+(D_{11}\Psi_1 + D_{12}\Psi_2) + \Psi_2^+(D_{21}\Psi_1 + D_{22}\Psi_2)] + \frac{1}{2!}[\Psi_i^+ D_{ij} \Psi_j]^2 + \dots\right)$$

$$= \int d\Psi_1^+ d\Psi_1\, d\Psi_2^+ d\Psi_2\, \frac{1}{2!}\left[D_{11}D_{22}\Psi_1^+\Psi_1\Psi_2^+\Psi_2(1+(-1)^4) + D_{12}D_{21}\Psi_1^+\Psi_1\Psi_2^+\Psi_2((-1)^3+(-1)^3)\right]$$

$$= (D_{11}D_{22} - D_{12}D_{21}) \;\boxed{\equiv \det \hat{D}}$$

$$\Rightarrow \mathcal{Z}_F = \int \mathcal{D}\Psi^+ \mathcal{D}\Psi \exp\left[-\int_0^\beta d\tau \int d^3x\, \bar{\Psi}(-i\partial\!\!\!/ - im + i\mu\gamma_4)\Psi\right]$$

$$\hookrightarrow \frac{1}{\sqrt{V}} \sum_{n,\vec{p}} e^{i(\vec{p}\cdot\vec{x} + \omega_n \tau)} \Psi_p$$

$$= \int \mathcal{D}\Psi^+ \mathcal{D}\Psi \exp\left\{\sum_{n,\vec{p}} \Psi_p^+ \underbrace{\beta[-\omega_n - i\mu + \gamma_4(-\vec{\gamma}\cdot\vec{p} + im)]}_{\equiv D_{\alpha\beta} \quad \alpha,\beta = 1-4} \Psi_p\right\}$$

$$= \det \hat{D}$$

determinant in Dirac-spinor space: $\det_{\alpha\beta} \beta \begin{pmatrix} (\omega_n + i\mu) + im & +i\vec{\sigma}\cdot\vec{p} \\ i\vec{\sigma}\cdot\vec{p} & (\omega_n + i\mu) - im \end{pmatrix}$

$$= \beta^2\left[(\omega_n + i\mu)^2 + m^2 + \vec{p}^{\,2}\right] \mathbb{1}_2$$

$$\Rightarrow \log \mathcal{Z}_F = \log(\det \hat{D}) = \operatorname{Tr} \log D = 2 \sum_{n,\vec{p}} \log\left\{\beta^2\left[(\omega_n + i\mu)^2 + \omega_p^2\right]\right\}$$

$$= 2 \sum_{\vec{p}} \sum_{n=1}^{\infty} \log\left\{\beta^4\left[(\omega_p + i\mu)^2 + \omega_n^2\right]\left[(\omega_p - i\mu)^2 + \omega_n^2\right]\right\}$$

$$\boxed{\log \mathcal{Z}_F = 2V\int\frac{d^3p}{(2\pi)^3}\left\{\beta\omega_p + \log\left(1+e^{-\beta(\omega_p-\mu)}\right) + \log\left(1+e^{-\beta(\omega_p+\mu)}\right)\right\}}$$

spin — anti-/particle zero-point — thermal particle — thermal antiparticle

Thermodynamics

$m,\mu \to 0$

$$\Omega = -\frac{T}{V}\log\mathcal{Z} = \varepsilon - \mu n - Ts \qquad -P_0 - 4\,\frac{7}{8}\,\frac{\pi^2}{90}T^4$$

$$s = -\frac{\partial\Omega}{\partial T} = -\int\frac{d^3p}{(2\pi)^3}\left[(1-f_+^F)\log(1-f_+^F) + f_+^F\log f_+^F + (\mu\to-\mu)\right] \qquad 4\,\frac{7}{8}\,\frac{4\pi^2}{90}T^3$$

$$\varepsilon-\mu n = \frac{\partial(\beta\Omega)}{\partial\beta} = \varepsilon_0 + \int\frac{d^3p}{(2\pi)^3}\left[(\omega_p-\mu)f_+^F + (\omega_p+\mu)f_-^F\right] \qquad \varepsilon_0 + 4\,\frac{7}{8}\,\frac{\pi^2}{30}T^4$$

$$f_+^F = \left(e^{-\beta(\omega_p-\mu)}+1\right)^{-1}$$

i.e. $n = \int\frac{d^3p}{(2\pi)^3}\left(f_+^F - f_-^F\right)$

Formal Notation

$$\boxed{\begin{aligned}\log\mathcal{Z}_B &= \log\left(\det\hat{D}_{KG}\right)^{-\frac{1}{2}} = -\frac{1}{2}\,\mathrm{Tr}\,\log\left(G_B^{-1}\right)\\ \log\mathcal{Z}_F &= \log\left(\det\hat{D}\right) = \mathrm{Tr}\,\log\left(G_F^{-1}\right)\end{aligned}}$$

inverse propagator

2.3 Interactions: Mean-Field Approximation (MFA) + Ground-State Condensates

Nontrivial groundstates for $T \to 0$,
(composite) fields develop finite expectation value:

- spontaneous magnetization $\langle M \rangle (T<T_c) \neq 0$
- "Bose" condensates from Fermion pairing
  - Cooper pairs $\langle ee \rangle (T<T_c) \neq 0$ (BCS)
  - Chiral Condensate in the QCD vacuum

    $\rightarrow$ scalar quark-antiquark pairing $\langle \bar{q}q \rangle (T<T_c) \neq 0$

e.g. $\mathcal{L}_{eff} = \bar{\Psi}(i\not{\partial} - m)\Psi + G(\bar{\Psi}\Psi)^2$

effective 4-quark interaction

assume presence of a "mean field" $x_0 \equiv \langle 0|\bar{\Psi}\Psi|0\rangle$
and expand around it: $\bar{\Psi}\Psi = x_0 + \delta(\bar{\Psi}\Psi)$

$\Rightarrow$ linearize interaction term:

$$G(\bar{\Psi}\Psi)^2 \simeq G\left(x_0^2 + 2x_0\,\delta(\bar{\Psi}\Psi)\right) = 2G x_0 \bar{\Psi}\Psi - G x_0^2$$

$$\Rightarrow \mathcal{L}_{eff}^{x_0} = \bar{\Psi}\,(i\not{\partial} - \underbrace{[m - 2Gx_0]}_{\equiv M^*\text{: effective mass}})\,\Psi - G x_0^2$$

$$= \bar{\Psi}\,(G_F^{eff})^{-1}\,\Psi - (M^* - m)^2/4G$$

$\Rightarrow$ thermodynamic potential

$$\boxed{\Omega} = -\frac{T}{V} \log \mathcal{Z}$$

$$= -\frac{T}{V} \log \left[\mathrm{Tr}\, \exp\left\{ \int d^4x\, \bar{\Psi}(-i\partial\!\!\!/ - iM^* + i\mu\gamma_4)\Psi - \beta V \frac{(M^*-m)^2}{4G} \right\} \right]$$

$$= -\frac{T}{V} \log \left[\det D^{nn'}_{\alpha\beta} \right] + \frac{(M^*-m)^2}{4G}$$

$$= -\frac{T}{V} \mathrm{Tr} \log \left[G_F(M^*)^{-1} \right] + \frac{(M^*-m)^2}{4G}$$

$m \to 0$
$\mu \to 0$
$T \to 0$

$$\boxed{= -2 N_c N_f \int \frac{d^3p}{(2\pi)^3} \sqrt{\vec{p}^{\,2} + M^{*2}} + \frac{M^{*2}}{4G}}$$

groundstate: $\frac{\partial \Omega}{\partial M^*} \stackrel{!}{=} 0$

$$\Rightarrow \boxed{M^* = 4 N_c N_f G \int \frac{d^3p}{(2\pi)^3} \frac{M^*}{\sqrt{\vec{p}^{\,2} + M^{*2}}}}$$

"Gap Equation" for Spontaneous Breaking of Chiral Symmetry (SBCS)

Solutions:

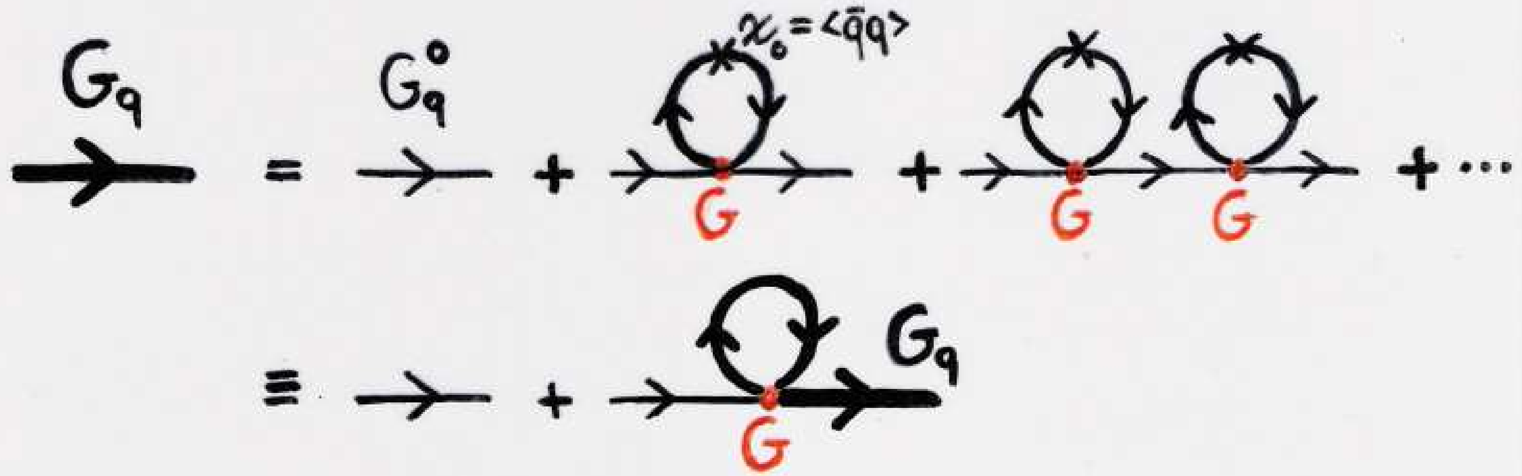
(i) $M^* = 0$

(ii) $M^* > 0$ for sufficiently large G

Alternative Derivation of Gap Equation:

Dyson Equation

Full Propagator ≙ iterated "selfenergy" due to interactions with (would-be) condensate



$$G_q = G_q^0 + G_q^0 \underbrace{G \chi_0}_{\equiv \Sigma \text{ (selfenergy)}} G_q \quad , \quad G_q^0 = \frac{1}{\not{p} - m_q + i\eta}$$

solve for G_q:

$$(1 - G_q^0 \Sigma)\, G_q = G_q^0 \quad \Rightarrow \quad G_q = \frac{1}{[(G_q^0)^{-1} - \Sigma]}$$

selfconsistency:

$$\boxed{\Sigma = G \langle \bar{q} q \rangle = G \,\mathrm{Tr}\, G_q = G \int \frac{d^4p}{(2\pi)^4} \,\mathrm{tr}\, G_q(p)}$$