Last time:
Derivative is a rate of change (or local slope)

In general,
\[
\frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}
\]

Simple Result:
If \( f(x) = c x^n \), then
\[
\frac{df}{dx} = c \cdot n \cdot x^{n-1}
\]

Today: Integrals
1) Define Integral by Fundamental thm. of Calculus
2) Simple formula for \( f(x) = c x^n \)
3) What is an integral conceptually?
4) What is application of calculus to Mechanics course?
5) Examples
6) Quiz
1. Fundamental Theorem of Calculus

Integration is Anti-Differentiation

\[
\frac{d}{dx} \left( \int f(x) \, dx \right) = f(x)
\]

Integral

For \( f(x) = cx^n \)

\[
\frac{d}{dx} f(x) = cnx^{n-1}
\]

\[
\int cx^n \, dx = \frac{c x^{n+1}}{n+1}
\]

Simple Formula for \( f(x) = cx^n \)

\[
\frac{d}{dx} \left[ \int cx^n \, dx \right] = \frac{d}{dx} \left[ \frac{c x^{n+1}}{n+1} \right]
\]

\[
= c \frac{n+1}{n+1} x^n = cx^n
\]
Indefinite Integral

\[ \int f(x) \, dx = \int kx^n \, dx \]

\[ = \frac{k}{n+1} x^{n+1} + C \]

\[ \frac{d}{dx} \left[ \frac{k}{n+1} x^{n+1} + C \right] = kx^n + 0 \]

C can be fixed by initial conditions of problem.
Definite Integral \[ \int_{x_1}^{x_2} f(x) \, dx \] gives constant result

\[ \int_{x_1}^{x_2} kx^n \, dx = \left( \frac{k}{n+1} x^{n+1} \right) \bigg|_{x_1}^{x_2} = \left( \frac{k}{n+1} x_2^{n+1} + C \right) - \left( \frac{k}{n+1} x_1^{n+1} + C \right) \]
Integral is area under the curve $f(x)$

\[\int_{x_1}^{x_2} f(x) \, dx\]

Examples: $f(x) = 3$

Area under curve from $x=1$ to $x=4$ is $L.H = 3 \cdot 3 = 9$

\[\int_{1}^{4} 3 \, dx = \left. \frac{3}{1} x \right|_{1}^{4} = 12 - 3 \]

\[= 9\]
\[ f(x) = x + 2 \]

\[
\int_0^4 f(x) \, dx
\]

\[
\text{Area} = 2 \cdot 4 + \frac{1}{2} \cdot 4 \cdot 4
\]

\[
= 8 + 8 = 16
\]

\[
\int_0^4 (x+2) \, dx = \left. \frac{x^2}{2} + 2x \right|_0^4
\]

\[
= \frac{16}{2} + 2(4) - (0 + 0) = 16
\]
Practice Integrals:

\( f(x) = x^2 \)

1. \( \int_0^5 f(x) \, dx = \int_0^5 x^2 \, dx \)
   \[ = \left. \frac{x^3}{3} \right|_0^5 \]
   \[ = \frac{5^3}{3} - 0 = \frac{125}{3} \]

2. \( \int_0^5 \frac{1}{x^2} \, dx = \int_0^5 x^{-2} \, dx \)
   \[ = \left. -\frac{1}{x} \right|_0^5 \]
   \[ = -\frac{1}{5} - 0 \]
   \[ = -\frac{1}{5} \] (divide by 0)

3. \( \int_1^5 \frac{1}{x^2} \, dx = \text{undefined} \)
   \[ = \left. -\frac{1}{x} \right|_1^5 \]
   \[ = \frac{-1}{5} - (-1) \]
   \[ = \frac{4}{5} \]
\[ f(x) = -x^2 \]
\[ \int_{0}^{5} -x^2 \, dx = -\frac{x^3}{3} \bigg|_{0}^{5} \]
\[ = -\frac{125}{3} - 0 \]
\[ = -\frac{125}{3} \]

Area "under" curve between \( x \)-axis and \( f(x) \)
\[
\int_0^1 \frac{\frac{d}{dx} \left( x^2 + 2x + 3 \right)}{f(x)} \, dx = ?
\]
\[
= \left[ x^2 + 2x + 3 \right]_0^1 = 1 + 2 + 3 - \cancel{(0 + 0 + 3)}
\]
\[
= 3
\]
\[
\frac{d}{dx} \left( x^2 + 2x + 3 \right) = 2x + 2
\]
\[
\int_0^1 (2x + 2) \, dx = \left[ x^2 + 2x \right]_0^1
\]
\[
= 1 + 2 - 0 = \frac{3}{3}
\]
Find the constant of integration if \( f(x) = x^2 \) and \( \int f(x) \, dx \) is 25 at \( x = 5 \).

\[
\int x^2 \, dx = \frac{x^3}{3} + C
\]

\[
\frac{5^3}{3} + C = 25
\]

\[
C = 25 - \frac{125}{3} = -\frac{50}{3}
\]
Calculus in Mechanics Course

\[ v = \frac{dx}{dt} \]

If given velocity,
\[ x(t) = \int v(t) \, dt \]
\[ = f(t) + C \]
\[ x(0) = C \]

Given \( a = -g \) constant
Initially at rest at height \( H \)
\[ v(0) = 0 \]
\[ y(0) = H \]

\[ v(t) = \int a \, dt = \int -g \, dt \]
\[ = -gt + C \]
\[ v(0) = -g(0) + C = C = v(0) = 0 \]
\[ v(t) = -gt \]

\[ y(t) = \int v(t) \, dt \]
\[ = \int (-gt) \, dt = -\frac{g}{2} t^2 + C_1 \]
\[ y(0) = -\frac{g}{2} 0^2 + C_1 = H \Rightarrow C_1 = H \]
\[ y(t) = -\frac{g}{2} t^2 + H \]
\[ f(x) = ax^2 + bx + c \]

1. Calculate \( \frac{df}{dx} \) using

\[
\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]

(check with simple formula)

2. Calculate \( \int f(x) \, dx \)