Last time:

Derivative is a rate of change (or local slope)

In general,

$$\frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Simple Result:

If $f(x) = cx^n$, then

$$\frac{df}{dx} = c \cdot n \cdot x^{n-1}$$

Today: Integrals

1) Define Integral by Fundamental thm. of Calculus
2) Simple formula for $f(x) = cx^n$
3) What is an integral conceptually?
4) What is application of calculus to Mechanics course
5) Examples
6) Quiz

1. Fundamental Theorem of Calculus

Integration is Anti-Differentiation

$$\frac{d}{dx}\left(\int f(x)\,dx\right) = f(x)$$

Integral

For $f(x) = cx^n$

$$\frac{d}{dx} f(x) = cnx^{n-1}$$

$$\boxed{\int \underline{cx^n\,dx} = \frac{cx^{n+1}}{n+1}}$$

Simple Formula for $f(x)=cx^n$

$$\frac{d}{dx}\left[\int cx^n\,dx\right] = \frac{d}{dx}\left[\frac{cx^{n+1}}{n+1}\right]$$

$$= c\,\frac{\cancel{n+1}}{\cancel{n+1}}\,x^n = cx^n \checkmark$$

Indefinite Integral

$f(x) = kx^n$

$$\int f(x)\,dx = \int \underline{kx^n}\,dx$$

$$= \frac{k}{n+1} x^{n+1} + C$$

$$\frac{d}{dx}\left[\frac{k}{n+1} x^{n+1} + C\right] = kx^n + \cancel{0}$$

C can be fixed by initial conditions of problem.

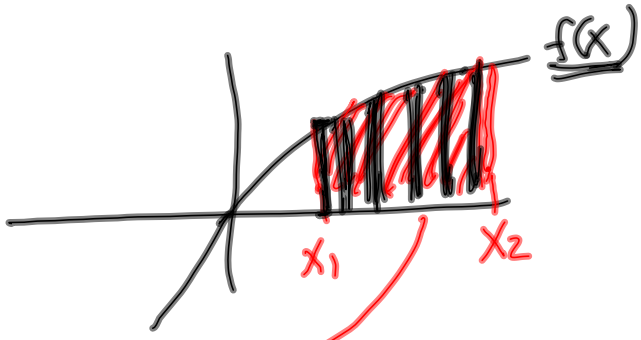
Definite Integral $\rightarrow$ gives constant result

$$\int_{x_1}^{x_2} f(x)\,dx$$

$$\int_{x_1}^{x_2} kx^n\,dx = \left(\frac{k}{n+1}x^{n+1}+C\right)\Bigg|_{x_1}^{x_2}$$

$$= \left(\frac{k}{n+1}x_2^{n+1}+\cancel{C}\right) - \left(\frac{k}{n+1}x_1^{n+1}+\cancel{C}\right)$$

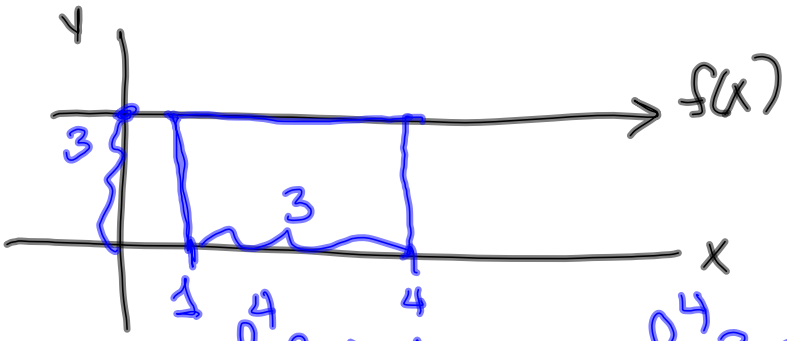
Integral is area under the curve $f(x)$



$$\int_{x_1}^{x_2} \underbrace{f(x)}_{\text{height}} \underbrace{dx}_{\text{width}}$$

Sum

Examples: $f(x) = 3$



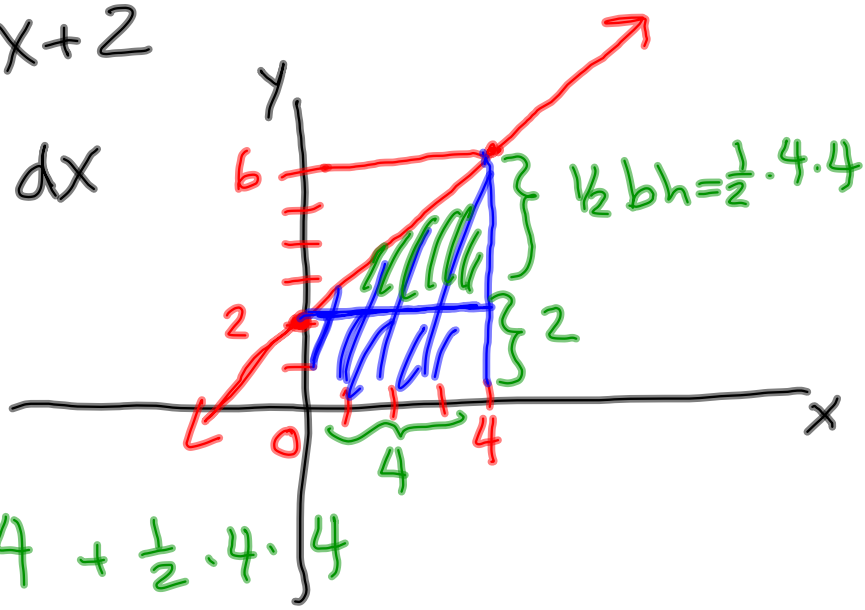
$$\int_1^4 f(x)\,dx = \int_1^4 3\,dx$$

Area under curve from $x=1$ to $x=4$ is $L \cdot H = 3 \cdot 3 = 9$

$$\int_1^4 3\,dx = \frac{3x^1}{1}\Big|_1^4 = 12 - 3 = 9$$

$f(x) = x+2$

$\int_0^4 f(x)\,dx$



$\text{Area} = 2 \cdot 4 + \frac{1}{2} \cdot 4 \cdot 4$

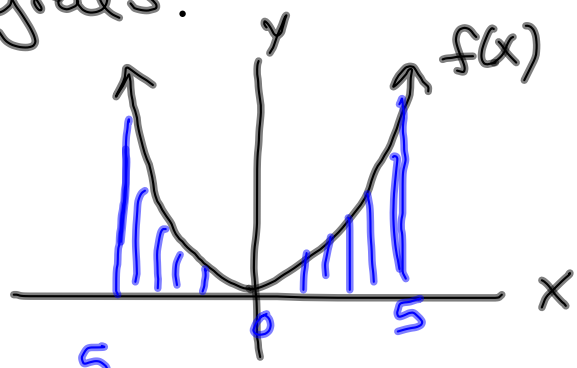
$= 8 + 8 = 16$

$\int_0^4 (x+2)\,dx = \frac{x^2}{2} + 2x \Big|_0^4$

$\frac{16}{2} + 2(4) - (0 + 0) = 16$

Practice Integrals:

$f(x) = x^2$



1. $\int_0^5 f(x)dx = \int_0^5 x^2\,dx$

$= \left.\frac{x^3}{3}\right|_0^5$

$= \frac{5^3}{3} - 0 = \frac{125}{3}$

2. $\int_0^5 \frac{1}{x^2}\,dx = \int_0^5 x^{-2}\,dx$

$= -x^{-1}\Big|_0^5 = -\frac{1}{x}\Big|_0^5$

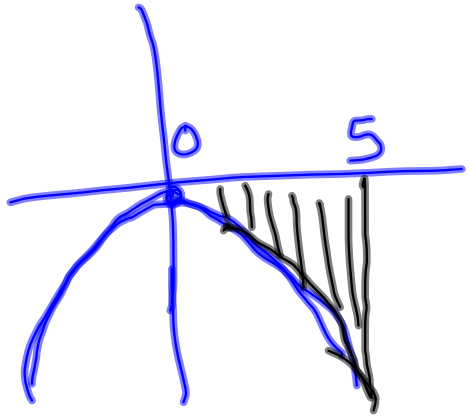
$= -\frac{1}{5} - \infty$

divide by 0

undefined

3. $\int_1^5 \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^5 = -\frac{1}{5} - (-1)$

$= 4/5$



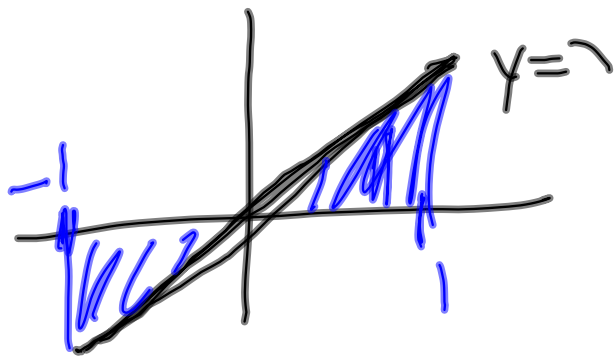
$f(x) = -x^2$

$$\int_0^5 -x^2 \, dx$$

$$= -\frac{x^3}{3}\Big|_0^5$$

$$= -\frac{125}{3} - 0$$

$$= -\frac{125}{3}$$



Area "under" curve btw. $\underline{\text{x-axis}}$ and $f(x)$

(dx)

$$\int_0^1 \frac{d}{dx}\{\underbrace{x^2+2x+3}_{f(x)}\}\,dx = ?$$

$$= \left[x^2+2x+3\right]_0^1 = 1+2+\cancel{3} - (0+0+\cancel{3})$$

$$= 3$$

$$\frac{d}{dx}\{x^2+2x+3\} = 2x+2$$

$$\int_0^1 (2x+2)\,dx = x^2+2x\Big|_0^1$$

$$= 1+2-0$$

$$= 3$$

Find the constant of integration if $f(x) = x^2$ and $\int f(x)\,dx$ is 25. at $x = 5$.

$$\int x^2\,dx = \frac{x^3}{3} + C$$

$$\frac{5^3}{3} + C = 25$$

$$C = 25 - \frac{125}{3} = \frac{-50}{3}$$

Calculus in Mechanics Course

$$v = \frac{dx}{dt}$$

If given velocity,

$$x(t) = \int v(t)\,dt$$

$$= f(t) + \underline{\underline{C}}$$

$$x(0) = C$$

Given $a = -g$ constant

Initially $(t=0)$ at rest at height H

$v(0) = 0$ $\qquad$ $y(0) = H$

$$v(t) = \int a\,dt = \int -g\,dt$$

$$= -gt + C$$

$$v(0) = \cancel{-g(0)}^{\,0} + C = C = v(0) = 0$$

$$v(t) = -gt$$

$$y(t) = \int v(t)\,dt$$

$$= \int (-gt)\,dt = -\frac{g}{2}t^2 + C_1$$

$$y(0) = \cancel{-\frac{g}{2}0^2} + C_1 = H \Rightarrow C_1 = H$$

$$y(t) = -\frac{g}{2}t^2 + H$$

$$f(x) = ax^2 + bx + c$$, a & b & c are constants

1. Calculate $\frac{df}{dx}$ using

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

(check with simple formula)

2. Calculate $\int f(x)\,dx$