Chapter 7 – The Work-Energy theorem

- Define work
  - Dot Product
- Examples of Calculating work
- Define Kinetic Energy

→ Work-Energy Theorem

Work

$W \sim \vec{F} \cdot \vec{d}$ "Dot Product"

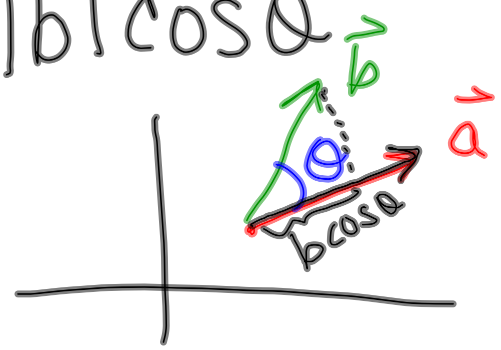
$$W = \int \vec{F} \cdot d\vec{r}$$

$$d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$$

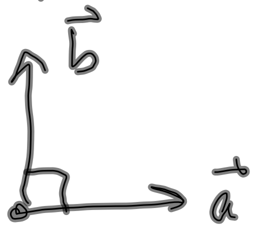
Define Dot Product:

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

θ is angle btw. $\vec{a}$ & $\vec{b}$



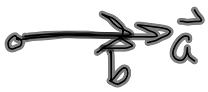
$\Rightarrow$ "length of $\vec{a}$ times the part of $\vec{b}$ that is parallel to $\vec{a}$"



for $\theta = 90°$, $\cos\theta = 0$

$$\vec{a} \cdot \vec{b} = 0$$

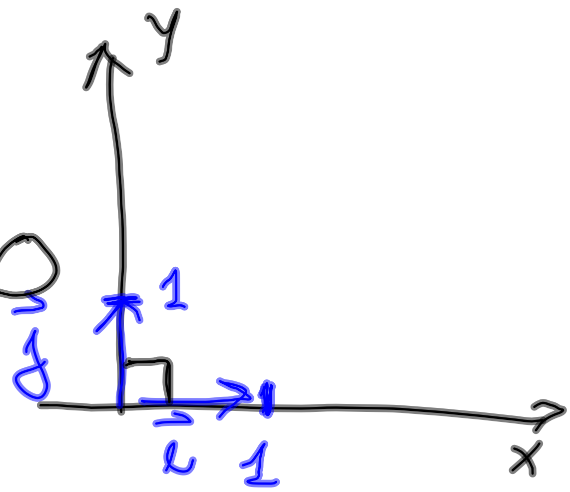
for $\theta = 0°$, $\cos\theta = 1$



$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|$$

$$\vec{i} \cdot \vec{i} = 1$$

$$\vec{i} \cdot \vec{j} = 1 \cdot 1 \cdot \cancel{\cos 90^\circ} = 0$$



$$\vec{a} = a_x \vec{i} + a_y \vec{j}$$

$$\vec{b} = b_x \vec{i} + b_y \vec{j}$$

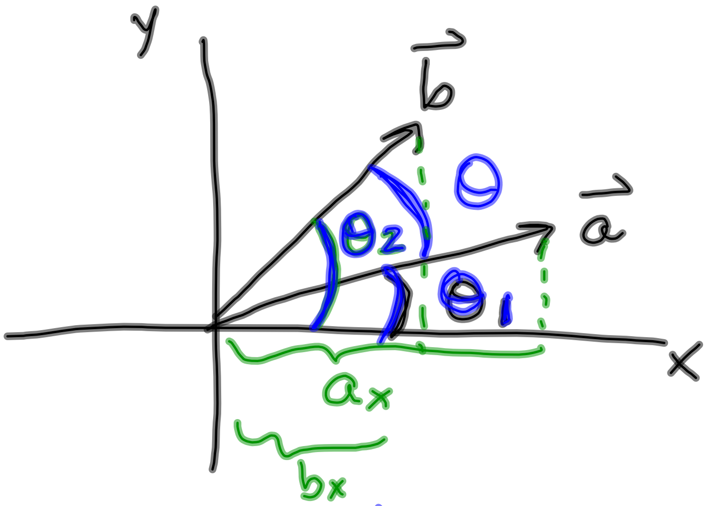
$$\vec{a} \cdot \vec{b} = (a_x \vec{i} + a_y \vec{j}) \cdot (b_x \vec{i} + b_y \vec{j})$$

$$a_x b_x \underbrace{(\vec{i} \cdot \vec{i})}_{1} + a_x b_y \underbrace{(\vec{i} \cdot \vec{j})}_{0} + a_y b_x \underbrace{(\vec{j} \cdot \vec{i})}_{0} + a_y b_y \underbrace{(\vec{j} \cdot \vec{j})}_{1}$$

$$= a_x b_x + a_y b_y$$

$$\vec{a} \cdot \vec{b} = \underline{|\vec{a}||\vec{b}|} \cos\theta = \underline{a_x b_x + a_y b_y}$$

$$\theta = \theta_2 - \theta_1$$



$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta = ab\cos\theta$$

$$= |\vec{a}||\vec{b}|\cos(\theta_2 - \theta_1)$$

$$\cos(\theta_2 - \theta_1) = \cos\theta_2\cos\theta_1 + \sin\theta_2\sin\theta_1$$

$$\rightarrow \vec{a}\cdot\vec{b} = ab(\cos\theta_2\cos\theta_1 + \sin\theta_2\sin\theta_1)$$

$$= \underbrace{a\cos\theta_1}_{a_x}\ \underbrace{b\cos\theta_2}_{b_x} + \underbrace{a\sin\theta_1}_{a_y}\ \underbrace{b\sin\theta_2}_{b_y}$$

$$\vec{a}\cdot\vec{b} = \underbrace{a_x b_x + a_y b_y}$$

$$= |\vec{a}||\vec{b}|\cos\theta$$

Defn. of Work:

"Work done by $\vec{F}$ on an object as that object moves from $\vec{r}_1$ to $\vec{r}_2$"

$$\equiv W^{\vec{F}}_{\vec{r}_1,\vec{r}_2} \equiv \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot d\vec{r}$$

$$d\vec{r} = dx\,\vec{\imath} + dy\,\vec{\jmath}$$

Integral over path from $\vec{r}_1$ to $\vec{r}_2$

If $\vec{F}$ is constant (does not depend on $\vec{r}$)

$$W^{\vec{F}}_{\vec{r}_1,\vec{r}_2} = \vec{F}\cdot\int_{\vec{r}_1}^{\vec{r}_2} d\vec{r} \qquad d\vec{r} = \underline{dx\,\vec{\imath} + dy\,\vec{\jmath}}$$

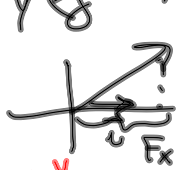
$$= \underbrace{\vec{F}\cdot\left[\int_{x_1 y_1}^{x_2 y_2} \vec{\imath}\,dx + \int_{x_1 y_1}^{x_2 y_2} \vec{\jmath}\,dy\right]}$$

F_x F_y

$\vec{r}_1 = (x_1, y_1)$ & $\vec{r}_2 = (x_2, y_2)$

$$\boxed{\vec{F} = \left(F_x\,\vec{\imath} + F_y\,\vec{\jmath}\right)} \cdot \left(\vec{\imath}\,dx + \vec{\jmath}\,dx\right)$$

$$F_x(\vec{\imath}\cdot\vec{\imath}) + F_y(\cancel{\vec{\jmath}\cdot\vec{\imath}})$$

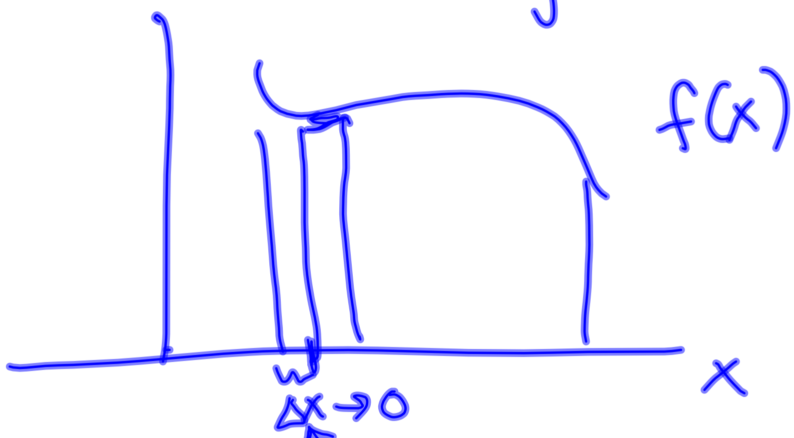


$$W^{\vec{F}\,(\text{constant})}_{\vec{r}_1,\vec{r}_2} = F_x \int_{x_1}^{x_2} dx + F_y \int_{y_1}^{y_2} dy$$

If $\vec{F}$ is not constant

$$\boxed{W^{\vec{F}}_{\vec{r}_1,\vec{r}_2} = \int_{x_1}^{x_2} F_x\,dx + \int_{y_1}^{y_2} F_y\,dy}$$

Remember Integrals:



$$\int f(x)\,dx = \lim_{\Delta x \to 0} f(x)\Delta x$$

$$W^{\vec{F}}_{\vec{r}_1,\vec{r}_2} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot d\vec{r} = \int_{x_1}^{x_2} F_x\,dx + \int_{y_1}^{y_2} F_y\,dy$$

"the bottom line"

Example of work done by a variable force:

$$\vec{F} = (8x - 16)\vec{i}$$

$F_x = 8x - 16$ and $F_y = 0$

calculate the work done by that force on an object as it moves from $x=0$ to $x=8$

$$W^{\vec{F}}_{x=0,8} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} \qquad d\vec{r} = dx\vec{i} + dy\vec{j}$$

$$= \int_{x_1}^{x_2} F_x dx + \int_{y_1}^{y_2} \cancel{F_y}^{\,0} dy$$

$$= \int_0^8 (8x - 16) dx$$

$$= \left[4x^2 - 16x\right]_0^8 = 256 - 128 - [0]$$

$$= 128 \text{ J}$$

Units:

$$[W] = [F][x] = N \cdot m = J$$

"Joule"

What is the work done by the force as object moves from $y=0$ to $y=8$?

$$F_x = 8x - 16 \qquad F_y = 0$$

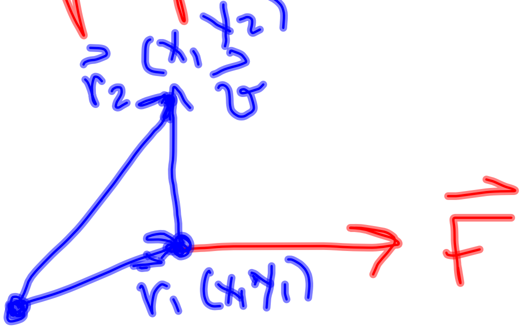
$$W^{\vec{F}}_{y=0,8} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} = \int_0^0 F_x\, dx + \int_0^8 \cancel{F_y}^{\,0}\, dy$$

$$= \int_0^0 (8x-16)\, dx$$

$$= \left[4x^2 - 16x \right]_0^0 = 0$$

same

$\therefore$ Force does no work for object's motion in direction perpendicular to force

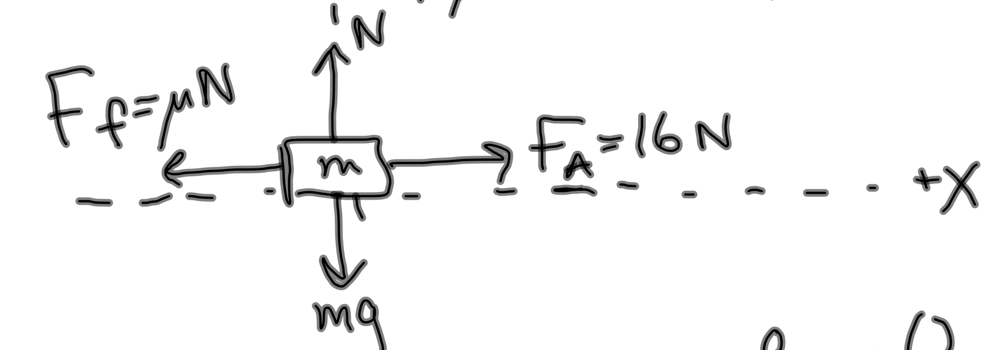


$$W^{F}_{r_1, r_2} = 0$$

$$\vec{F} \perp d\vec{r}$$

Example:

A block of mass $m = 2.5\,kg$ pushed along a horizontal table with a constant horizontal force of 16N. Coefficient of friction btw. block + table is 0.2. Calculate the work done by each force on the block as the block moves <u>2.2m</u>.



$$W^{F_A}_{x=0,2.2} = \int_0^{2.2} \underline{F_{Ax}}\,dx + \cancel{\int_0^0 F_{Ay}\,dy} \rightarrow 0$$

$$= 16\int_0^{2.2} dx = 16 \times \Big|_0^{2.2}$$

$$= 16(2.2-0) = \boxed{35.2\,J}$$

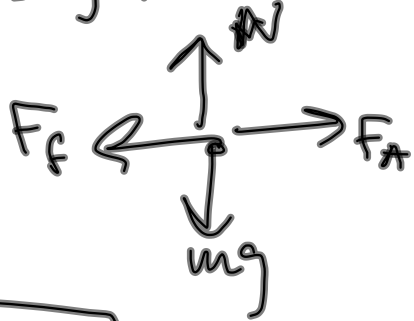
$$W^{F_f}_{x=0,2.2} = \int_0^{2.2} \underbrace{F_{fx}}_{=F_f}\,dx + \cancel{\int_0^{0} F_{fy}\,dy}^{\,0}$$

$$\sum F_y = N - mg = 0$$

$$N = mg = (2.5\,kg)(9.8\,\tfrac{m}{s^2})$$

$$F_f = \mu N = (0.2)(2.5\,kg)(9.8\,m/s^2)$$

$$\boxed{F_{fx} = F_f}$$



$$W^{F_f}_{x=0,2.2} = -\overbrace{(0.2)(2.5\,kg)(9.8\,\tfrac{m}{s^2})}^{F_f}(\underbrace{2.2m-0}_{x_2-x_1})$$

$$\boxed{= -10.8\,J}$$

$$W^{N}_{x=0,2.2} = \int \cancel{N_x}^{0}\, dx + \int_0^0 N_y^{0}\, dy$$

$$= 0$$

$$W^{F_g}_{x=0,2.2} = \int_0^{2.2} \cancel{F_{gx}}^{0}\, dx + \int_0^0 F_{gy}^{0}\, dx$$

$$= 0$$

$$W^{total} = W^{F_A} + W^{F_f} + W^{N} + W^{F_g}$$

$$= 24.4\,\mathrm{J}$$

$\boxed{\text{Kinetic Energy}}$ – energy of a moving object

$$K.E. = \frac{1}{2} m v^2 \rightarrow \text{scalar}$$

speed
magnitude of velocity

$$[K.E] = [m][v]^2 = kg \cdot \frac{m^2}{s^2}$$

$$= \Big(\underbrace{kg \cdot \frac{m}{s^2}}_{N}\Big) m = J$$

Example:

$m = 3kg$

$\vec{v} = (6\vec{i} - 2\vec{j}) \frac{m}{s}$

$$K.E = \frac{1}{2} m v^2 = \frac{1}{2} m (\vec{v} \cdot \vec{v})$$

$$= \frac{1}{2}(3kg)\underbrace{(6\vec{i} - 2\vec{j}) \cdot (6\vec{i} - 2\vec{j})}_{6^2 + 2^2}$$

$$= 60 J$$

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{6^2 + 2^2}$$

$$\boxed{\begin{aligned} W^{total} &= KE_f - KE_i \\ &= \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2 \end{aligned}}$$

Work-Energy theorem

Typical Problem:
Given forces & initial + final positions, find $\Delta KE = KE_f - KE_i$

$\uparrow$ change