Solve using Newtonian Mechanics
1.) Assuming a circular loop of radius $R$, and taking the approximation that the person’s pedaling just compensates for friction,
(a) find the expression for the minimum height $h'$ needed at the top of the ramp.
(b) What is the person’s horizontal component of velocity on the ramp as a function of his horizontal distance $s$ from the bottom, given that the ramp height is described by a known function $h(s)$?
(c) Starting from this expression, find the time taken to reach the bottom of the ramp if $h(s) = as$, where $a$ is a constant.

Alternate Form of Euler Equation
2.) If $f = f(y, y_x, x)$ (where $y_x$ denotes $dy/dx$), show the equivalence of the two forms of Euler’s equation:

\[ \frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y_x} = 0 \]

and

\[ \frac{\partial f}{\partial x} - \frac{d}{dx} \left( f - y_x \frac{\partial f}{\partial y_x} \right) = 0. \]

3.) Show that the condition that

\[ J = \int f(x, y) dx \]

have a stationary value
(a) leads to $f(x, y)$ independent of $y$ and
(b) yields no information about any $x$-dependence.
We get no (continuous, differentiable) solution. To be a meaningful variational problem, dependence on $y_x$ or higher derivatives is essential.
a.) 
\[ mgh' = \frac{1}{2}mv_{top}^2 + mg(2R) \]

\[ \Rightarrow h' = \frac{1}{2}\frac{v_{top}^2}{g} + 2R \]

\[ \sum F_{top} = -F_N - mg = -\frac{mv_{top}^2}{R} \]

\[ \Rightarrow v_{top}^2 = Rg \]

\[ \Rightarrow h' = \frac{R}{2} + 2R = \frac{5}{2}R \]

b.) 
\[ \frac{mv^2}{2} = mgh' - mgh \]

\[ \Rightarrow v^2 = 2g(h' - h) = \left(\frac{ds}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2 \]

h(s) is known \[ \Rightarrow \frac{dh}{dt} = \frac{dh}{ds} \frac{ds}{dt} \]

\[ \Rightarrow 2g(h' - h) = \left(\frac{ds}{dt}\right)^2 \left[1 + \left(\frac{dh}{ds}\right)^2\right] \]

\[ \Rightarrow \frac{ds}{dt} = \sqrt{\frac{2g(h' - h)}{1 + \left(\frac{dh}{ds}\right)^2}} \]

\[ \Rightarrow \frac{dh}{ds} = \alpha \Rightarrow \frac{ds}{dt} = -\sqrt{\frac{2g(h' - h)}{1 + \alpha^2}} \]

c.) h(s) = \text{as}
\[ h(t=0) = h' \Rightarrow s = \frac{h'}{\sqrt{2g}} \quad \text{At bottom} \quad h = 0 \Rightarrow s = 0 \]

\[
-\left( \frac{1+a^2}{2g(h'^2+a)} \right) ds = -\left( \int_{0}^{t} \frac{1+a^2}{2g(h'^2+a)} ds \right) = \int_{0}^{t} dt
\]

Let \( u = 2g(h'-as) \)
\[
du = -2gad\,ds
\]

\[
= \frac{\sqrt{1+a^2}}{2ga} \int_{0}^{2gh'} u^{-1/2} \, du
\]

\[
= \frac{\sqrt{1+a^2}}{ga} u^{1/2} \Bigg|_{0}^{2gh'} = t
\]

\[ \Rightarrow t = \sqrt{\frac{2h'(1+a^2)}{ga^2}} \]
2. $\frac{af}{ay} - \frac{d}{dx}\left(\frac{af}{ay_y} \right) = 0 \quad \text{Euler's Equation}$

\[ f = f(y, y_x, x) \]

\[ \frac{df}{dx} = \frac{af}{ay} y_x + \frac{af}{ay_y} y_{xx} + \frac{af}{ax} \]

\[ \Rightarrow \quad \frac{af}{ax} - \frac{d}{dx}\left(f - y_x \frac{af}{ax}\right) = 0 \]

\[ = \frac{af}{ax} - \frac{af}{ay} y_x - \frac{af}{ay_y} y_{xx} - \frac{af}{ax} + y_{xx} \frac{af}{ax} + y_x \frac{d}{dx}(\frac{af}{ax}) \]

\[ = -y_x \left( \frac{af}{ay} - \frac{d}{dx}(\frac{af}{ax}) \right) = 0 \]

3. If $J = \int f(x, y) \, dx$

\[ \frac{af}{ay} = 0 \quad \text{since} \quad \frac{af}{ay_y} = 0 \quad \text{and} \quad \frac{af}{ay} = \frac{d}{dx}(\frac{af}{ay_y}) = \frac{d}{dx}(0) \]

\[ \therefore f(x, y) \text{ is independent of } y. \]

\[ \Rightarrow \quad \text{yields no } x\text{-dependence.} \]
\( q_i = q_i(s_1, s_2, \ldots, s_n, t) \); \( L = L(q_i, s_i, t) = L(s_i, s_i, t) \)

Assume \( \frac{d}{dt} \left( \frac{\partial L}{\partial q_i} \right) - \frac{\partial L}{\partial q_i} = 0 \)

\[
\frac{\partial L}{\partial s_j} = \sum_k \frac{\partial L}{\partial q_k} \cdot \frac{\partial q_k}{\partial s_j}^0 + \frac{\partial L}{\partial q_k} \cdot \frac{\partial q_k}{\partial s_j}^0
\]

\[
\frac{\partial L}{\partial s_j} = \sum_k \frac{\partial L}{\partial q_k} \cdot \frac{\partial q_k}{\partial s_j}^0 + \frac{\partial L}{\partial q_k} \cdot \frac{\partial q_k}{\partial s_j}^0
\]

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial s_j} \right) = \sum_k \frac{d}{dt} \left( \frac{\partial L}{\partial q_k} \cdot \frac{\partial q_k}{\partial s_j}^0 \right) = \sum_k \left[ \frac{\partial L}{\partial q_k} \frac{d}{dt} \left( \frac{\partial q_k}{\partial s_j}^0 \right) \right]
\]

\[
\Rightarrow \frac{d}{dt} \left( \frac{\partial L}{\partial s_j} \right) - \frac{\partial L}{\partial s_j} = \sum_k \left[ \frac{\partial L}{\partial q_k} \frac{d}{dt} (\frac{\partial q_k}{\partial s_j})^0 \right. \left. + \frac{d}{dt} (\frac{\partial L}{\partial q_k} \cdot \frac{\partial q_k}{\partial s_j}^0) - \frac{\partial L}{\partial q_k} \cdot \frac{\partial q_k}{\partial s_j}^0 \right]
\]

\[
\sum_k \left[ \frac{\partial L}{\partial q_k} \frac{d}{dt} (\frac{\partial q_k}{\partial s_j})^0 \right. \left. + \frac{d}{dt} (\frac{\partial L}{\partial q_k}) \frac{\partial q_k}{\partial s_j}^0 \right. \left. - \frac{\partial L}{\partial q_k} \cdot \frac{\partial q_k}{\partial s_j} \right]
\]

\[
= \sum_k \left[ \frac{\partial L}{\partial q_k} \left( \frac{\partial q_k}{\partial s_j} \right)^0 \right. \left. + \frac{d}{dt} \left( \frac{\partial L}{\partial q_k} \right) \frac{\partial q_k}{\partial s_j} \right. \left. - \frac{\partial L}{\partial q_k} \cdot \frac{\partial q_k}{\partial s_j} \right]
\]

\[
= \sum_k \left( \frac{\partial q_k}{\partial s_j} \right)^0 \left[ \frac{d}{dt} \left( \frac{\partial L}{\partial q_k} \right) - \frac{\partial L}{\partial q_k} \right]
\]

\[
= 0
\]

Extra Credit 5 pts.