Physics 606 -- Problem Set 9

Due in class, Thursday, April 23, 2009

(This is the last problem set!)

Do the following problems from Merzbacher, Chapter 11:

Exercises 11.15, 11.16, and 11.20

For 11.20, construct P_3, instead of P_2.

End-of-chapter problems 2 and 4

Do the following problems from Merzbacher, Chapter 12:

Exercises 12.2 and 12.6

In addition, do the following, which is motivated by some questions you asked at the end of class on Wednesday, April 15:

(a) Construct the matrices for J_x, J_y, and J_z for the case $j = ½$.

(b) Construct the matrix associated with the measurement of J_n, where $\mathbf{n}$ is a unit vector in the (θ,φ) direction. [Note: θ (polar angle) and φ (azimuthal angle) are the standard spherical coordinates.]

(c) Assume the system is initially prepared in the state $|\,jm\,\rangle = |\,½\;½\,\rangle$. What is the expectation value for a measurement of J_n?

(d) What specific values of J_n can you get during the measurement described in part (c)? Find the probability to obtain each of those values. Verify that your results are consistent with your answer to part (c).

(e) Assume you've measured J_n and found the smallest possible value. If you then measure J_z, what specific values can you get? What is the probability to obtain each of those values?

Chapter 15, Exercise 11.15

$$L_z(x \pm iy)^m = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)(x \pm iy)^m$$

$$= \frac{\hbar}{i}\left[x\,m(x \pm iy)^{m-1}(\pm i) - y\,m(x \pm iy)^{m-1}(+1)\right]$$

$$= \hbar m(x \pm iy)^{m-1}\left[\pm x + iy\right] = \hbar m(x \pm iy)^{m-1}(\pm 1)(x \pm iy)$$

$$= (\pm m)\hbar(x \pm iy)^m \Rightarrow \underline{\underline{(x \pm iy)^m \text{ is an eigenvector}}} \text{ of } L_z$$

with eigenvalue $\pm m\hbar$.

<u>Note</u>: In spherical coordinates $(x \pm iy)^m = r^m(\sin\theta\cos\varphi \pm i\sin\theta\sin\varphi)^m$

$$= r^m\sin^m\theta(\cos\varphi \pm i\sin\varphi)^m = r^m\sin^m\theta\, e^{\pm im\varphi}, \text{ and}$$

$$L_z\left(r^m\sin^m\theta\, e^{\pm im\varphi}\right) = r^m\sin^m\theta\,\frac{\hbar}{i}\frac{\partial}{\partial\varphi}\left(e^{\pm im\varphi}\right)$$

$$= \hbar r^m\sin^m\theta\,\frac{\pm im}{i}e^{\pm im\varphi} = \pm m\hbar(x \pm iy)^m$$ gives the same answer.

Chapter 11, Exercise 11.16

Assume $l'>l$. Note that $P_l(\xi)=\frac{1}{2^l l!}\frac{d^l}{d\xi^l}(\xi^2-1)^l$ involves

the l^{th} derivative of a polynomial of degree $2l \Rightarrow P_l(\xi)$ is a polynomial of degree l.

Then $\int_{-1}^{1} P_l(\xi)\,P_{l'}(\xi)\,d\xi = \int_{-1}^{1} P_l(\xi)\frac{1}{2^{l'}l'!}\frac{d^{l'}}{d\xi^{l'}}(\xi^2-1)^{l'}d\xi$

$= \left[P_l(\xi)\frac{1}{2^{l'}l'!}\frac{d^{l'-1}}{d\xi^{l'-1}}(\xi^2-1)^{l'}\right]_{\xi=-1}^{1} - \int_{-1}^{1}\frac{dP_l(\xi)}{d\xi}\frac{1}{2^{l'}l'!}\frac{d^{l'-1}}{d\xi^{l'-1}}(\xi^2-1)^{l'}d\xi$

$\frac{d^{l'-1}}{d\xi^{l'-1}}\left[(\xi^2-1)^{l'}\right]$ has (ξ^2-1) as a factor. Thus the $[\ \ ]_{\xi=-1}^{1}$ is zero

at both limits, and we get:

$$\int_{-1}^{1} P_l(\xi)P_{l'}(\xi)\,d\xi = (-1)\int_{-1}^{1}\frac{dP_l(\xi)}{d\xi}\frac{1}{2^{l'}l'!}\frac{d^{l'-1}}{d\xi^{l'-1}}(\xi^2-1)^{l'}d\xi.$$

Repeat the integration by parts another $l'-1$ times, to find:

$$\int_{-1}^{1} P_l(\xi)P_{l'}(\xi)\,d\xi = (-1)^{l'}\frac{1}{2^{l'}l'!}\int_{-1}^{1}\frac{d^{l'}P_l(\xi)}{d\xi^{l'}}(\xi^2-1)^{l'}d\xi$$

But P_l is a polynomial of degree $l<l' \Rightarrow \frac{d^{l'}P_l(\xi)}{d\xi^{l'}}=0$, and

$$\int_{-1}^{1} P_l(\xi)P_{l'}(\xi)\,d\xi = 0 \quad ✓$$

Chapter 11, Exercise 11.20

We start from $1, \xi, \xi^2, \xi^3, \ldots$ for $-1 \le \xi \le +1$.

$1, \xi^2, \xi^4, \ldots$ are even fns of ξ, while $\xi, \xi^3, \xi^5, \ldots$ are odd fns of ξ. Every even fn is orthogonal to every odd fn when integrating over $[-1,+1]$, so the two sets don't mix.

$\therefore$, to construct P_3, start with $P_1(\xi) = \xi$, and $b(\xi) = \xi^3$.

$$\int_{-1}^{1} P_1(\xi)\, b(\xi)\, d\xi = \int_{-1}^{1} \xi^4\, d\xi = \left[\frac{\xi^5}{5}\right]_{\xi=-1}^{+1} = \frac{2}{5}.$$

Meanwhile, $\int_{-1}^{+1} |P_1(\xi)|^2\, d\xi = \int_{-1}^{1} \xi^2\, d\xi = \frac{2}{3}$.

Then $P_{3,\text{unnorm}}(\xi) = \xi^3 - P_1(\xi) \dfrac{\int_{-1}^{1} P_1(\xi)\, b(\xi)\, d\xi}{\int_{-1}^{1} P_1^2(\xi)\, d\xi}$ will be orthogonal to $P_1(\xi)$.

We find $P_{3,\text{unnorm}}(\xi) = \xi^3 - \xi \dfrac{\frac{2}{5}}{\frac{2}{3}} = \xi^3 - \frac{3}{5}\xi$

The standard normalization for the Legendre polynomials is $P_\ell(+1) = 1$.

$$P_{3,\text{unnorm}}(+1) = 1 - \frac{3}{5} = \frac{2}{5} \Rightarrow P_3(\xi) = \frac{5}{2}\left(\xi^3 - \frac{3}{5}\xi\right) = \boxed{\frac{1}{2}\left(5\xi^3 - 3\xi\right)}$$

Check: $\int_{-1}^{1} P_1(\xi) P_3(\xi)\, d\xi = \frac{1}{2}\int_{-1}^{1} (5\xi^4 - 3\xi^2)\, d\xi = \frac{1}{2}\left[\xi^5 - \xi^3\right]_{\xi=-1}^{+1} = \frac{1}{2}(0-0) = 0$ ✓

Chapter 11, Problem 2

Choose β so that $a = \sqrt{a^2+b^2}\cos\beta$, $b = \sqrt{a^2+b^2}\sin\beta$. Also, $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$.

Then $f(\vec{r}) = ax+by = \sqrt{a^2+b^2}\,(r\sin\theta)\left(\cos\beta\cos\varphi + \sin\beta\sin\varphi\right)$

Let $\sqrt{a^2+b^2}\,(r\sin\theta) = 2C(\theta)$. Then:

$$f(\vec{r}) = 2C(\theta)\left[\cos\beta\,\frac{e^{i\varphi}+e^{-i\varphi}}{2} + \sin\beta\,\frac{e^{i\varphi}-e^{-i\varphi}}{2i}\right]$$

$$= 2C(\theta)\left[\frac{\cos\beta - i\sin\beta}{2}e^{i\varphi} + \frac{\cos\beta+i\sin\beta}{2}e^{-i\varphi}\right] = C(\theta)\left(e^{-i\beta}e^{i\varphi} + e^{i\beta}e^{-i\varphi}\right)$$

$U_R = e^{-i\frac{\alpha L_z}{\hbar}}$. $L_z\left(C(\theta)e^{-i\beta}e^{i\varphi}\right) = \frac{\hbar\,\partial}{i\,\partial\varphi}\left(C(\theta)e^{-i\beta}e^{i\varphi}\right) = \hbar C(\theta)e^{-i\beta}e^{i\varphi}$

$$\Rightarrow U_R\left[C(\theta)e^{-i\beta}e^{i\varphi}\right] = e^{-i\frac{\alpha\hbar}{\hbar}}\left[C(\theta)e^{-i\beta}e^{i\varphi}\right] = C(\theta)e^{-i(\alpha+\beta)}e^{i\varphi}$$

Likewise, $L_z\left(C(\theta)e^{i\beta}e^{-i\varphi}\right) = \frac{\hbar}{i}\frac{\partial}{\partial\varphi}\left(C(\theta)e^{i\beta}e^{-i\varphi}\right) = -\hbar C(\theta)e^{i\beta}e^{-i\varphi}$

$$\Rightarrow U_R\left[C(\theta)e^{i\beta}e^{-i\varphi}\right] = e^{-i\frac{\alpha(-\hbar)}{\hbar}}C(\theta)e^{i\beta}e^{-i\varphi} = C(\theta)e^{i(\alpha+\beta)}e^{-i\varphi}$$

Putting it together, we get:

$$U_R f(\vec{r}) = C(\theta)\left(e^{-i(\alpha+\beta)}e^{i\varphi} + e^{i(\alpha+\beta)}e^{-i\varphi}\right)$$
$$= a'x + b'y$$

where $a' = \sqrt{a^2+b^2}\cos(\alpha+\beta)$, $b' = \sqrt{a^2+b^2}\sin(\alpha+\beta)$, and we have rotated $f(\vec{r})$ about z by α radians.

Chapter 11, Problem 4

$\frac{1}{r}\vec{r}\cdot\vec{p} = \frac{x}{r}p_x + \frac{y}{r}p_y + \frac{z}{r}p_z$. Its adjoint will be $\vec{p}\cdot\vec{r}\frac{1}{r}$

$$\vec{p}\cdot\vec{r}\frac{1}{r}f = \frac{\hbar}{i}\left[\frac{\partial}{\partial x}\left(\frac{xf}{r}\right) + \frac{\partial}{\partial y}\left(\frac{yf}{r}\right) + \frac{\partial}{\partial z}\left(\frac{zf}{r}\right)\right]$$

$$= \frac{\hbar}{i}\left[\left(\frac{\partial x}{\partial x}\right)\left(\frac{f}{r}\right) + x\left(\frac{\partial \frac{1}{r}}{\partial x}\right)f + \frac{x}{r}\frac{\partial f}{\partial x} + \cdots\right]$$

$$= \frac{\hbar}{i}\left[\frac{f}{r} + x\left[\frac{-\frac{1}{2}(2x)}{r^3}\right]f + \frac{x}{r}\frac{\partial f}{\partial x} + \cdots\right]$$

$$\Rightarrow \vec{p}\cdot\vec{r}\frac{1}{r}f = \frac{\hbar}{i}\left[3\frac{f}{r} - \frac{x^2+y^2+z^2}{r^3}f + \frac{x}{r}\frac{\partial f}{\partial x} + \frac{y}{r}\frac{\partial f}{\partial y} + \frac{z}{r}\frac{\partial f}{\partial z}\right]$$

$= \frac{\hbar}{i}\left(2\frac{f}{r}\right) + \frac{1}{r}\vec{r}\cdot\vec{p}f$. This is $\neq \frac{1}{r}\vec{r}\cdot\vec{p}$, so it's not hermitian.

Symmetrizing, we get $\frac{1}{2}\left[\frac{1}{r}\vec{r}\cdot\vec{p} + \vec{p}\cdot\vec{r}\frac{1}{r}\right] = \frac{1}{2}\left[\frac{1}{r}\vec{r}\cdot\vec{p} + \frac{2\hbar}{i}\frac{1}{r} + \frac{1}{r}\vec{r}\cdot\vec{p}\right]$

$= \frac{1}{r}\vec{r}\cdot\vec{p} + \frac{\hbar}{i}\frac{1}{r} = \frac{\hbar}{i}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)$, which is hermitian by construction

Meanwhile, $\frac{1}{2m}\left[\frac{\hbar}{i}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\right]^2 f = \frac{-\hbar^2}{2m}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)f$

$$= \frac{-\hbar^2}{2m}\left[\frac{\partial^2 f}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{f}{r}\right) + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}f\right] = \frac{-\hbar^2}{2m}\left[\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} - \frac{1}{r^2}f + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}f\right]$$

$$= \frac{-\hbar^2}{2m}\left[\frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r}\right] = -\frac{\hbar^2}{2mr^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right). \checkmark$$

Thus, by using the hermitian $p_r = \frac{\hbar}{i}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)$, we get the proper form for ∇^2 in spherical coordinates with:

$$H = \frac{p_r^2}{2m} + \frac{L^2}{2mr^2} + V(r)$$

Chapter 12, Exercise 12.2

$$j_\ell(\rho) \approx \frac{\cos\left(\rho - \frac{(\ell+1)\pi}{2}\right)}{\rho} \Rightarrow \frac{dj_\ell}{d\rho} \approx \frac{-\sin\left(\rho - \frac{(\ell+1)\pi}{2}\right)}{\rho} - \frac{\cos\left(\rho - \frac{(\ell+1)\pi}{2}\right)}{\rho^2}$$

$$\frac{d^2 j_\ell}{d\rho^2} \approx \frac{-\cos\left(\rho - \frac{(\ell+1)\pi}{2}\right)}{\rho} + \frac{\sin\left(\rho - \frac{(\ell+1)\pi}{2}\right)}{\rho^2} + \frac{\sin\left(\rho - \frac{(\ell+1)\pi}{2}\right)}{\rho^2} + \frac{2\cos\left(\rho - \frac{(\ell+1)\pi}{2}\right)}{\rho^3}$$

$$\Rightarrow \frac{d^2 j_\ell}{d\rho^2} + \frac{2}{\rho}\frac{dj_\ell}{d\rho} + \left(1 - \frac{\ell(\ell+1)}{\rho^2}\right) j_\ell$$

$$\approx \cancel{\frac{2\cos\left(\rho - \frac{(\ell+1)\pi}{2}\right)}{\rho^3}} + \cancel{\frac{2\sin\left(\rho - \frac{(\ell+1)\pi}{2}\right)}{\rho^2}} - \cancel{\frac{\cos\left(\rho - \frac{(\ell+1)\pi}{2}\right)}{\rho}}$$

$$- \cancel{\frac{2\cos\left(\rho - \frac{(\ell+1)\pi}{2}\right)}{\rho^3}} - \cancel{\frac{2\sin\left(\rho - \frac{(\ell+1)\pi}{2}\right)}{\rho^2}}$$

$$+ \cancel{\frac{\cos\left(\rho - \frac{(\ell+1)\pi}{2}\right)}{\rho}} - \frac{\ell(\ell+1)\cos\left(\rho - \frac{(\ell+1)\pi}{2}\right)}{\rho^3} = -\frac{\ell(\ell+1)\cos\left(\rho - \frac{(\ell+1)\pi}{2}\right)}{\rho^3}$$

The left-hand side is zero to $O\left(\frac{1}{\rho^2}\right)$. The first non-zero term is $O\left(\frac{1}{\rho^3}\right)$. ✓

Chapter 12, Exercise 12.6

Let $H = H_0 + V$ where $V(\vec{r}) \geq 0$ for all $\vec{r}$

Let $H\psi = E\psi$ be a normalized eigenfn of H with eigenvalue E.

Then $\langle\psi|H_0|\psi\rangle = \langle\psi|H-V|\psi\rangle = E - \langle\psi|V|\psi\rangle$ is an upper limit for an eigenvalue of H_0. But

$$\langle\psi|V|\psi\rangle = \int \psi^* V \psi \, d^3r = \int V(r) |\psi|^2 d^3r$$

$\int |\psi|^2 d^3r = 1$ and $V > 0$ where $|\psi|^2$ is non-zero. & Also $|\psi|^2 \geq 0 \Rightarrow$

$\int V(r) |\psi|^2 d^3r > 0 \Rightarrow \langle\psi|H_0|\psi\rangle < E$, and there must be an eigenstate of H_0 with a lower energy than E.

For a central potential, $H_\ell = \frac{p_r^2}{2m} + \frac{\ell(\ell+1)\hbar^2}{2mr^2} + V(r)$, and

$H_0 < H_1 < H_2 < H_3 < \cdots$ everywhere $\Rightarrow$

the lowest s state is below the lowest p state, which is below the lowest d state, which is below the lowest f state, ...,

until you run out of angular momenta with bound states.
(Above argument doesn't work for unbound states because we can't normalize ψ.)

Problem Set 9: Extra Problem

(a) ~~Order~~ The matrix indices are

$$\begin{array}{c|cc} & \frac{1}{2}\frac{1}{2} & \frac{1}{2}\,{-\frac{1}{2}} \\ \hline \frac{1}{2}\frac{1}{2} & - & - \\ \frac{1}{2}\,{-\frac{1}{2}} & - & - \end{array}$$

Then $\boxed{J_z = \hbar\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{-1}{2} \end{pmatrix}}$

$$\sqrt{\tfrac{1}{2}\left(\tfrac{3}{2}\right)-\tfrac{1}{2}\left(-\tfrac{1}{2}\right)} = \sqrt{\tfrac{3}{4}+\tfrac{1}{4}} = 1 \Rightarrow$$

$$J_+ = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad J_- = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$\Rightarrow$

$$J_x = \frac{J_+ + J_-}{2} = \boxed{\hbar\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}}$$

$$J_y = \frac{i}{2}\left(J_- - J_+\right) = \boxed{\hbar\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}}$$

(b) $$\hat{n} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} \Rightarrow J_n = \hat{n}\cdot\vec{J} = \boxed{\frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{pmatrix}}$$

(c) $$\langle J_n\rangle = \left\langle \tfrac{1}{2}\tfrac{1}{2}\right| J_n \left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} J_n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\cos\theta$$

(d) Ignore $\frac{\hbar}{2}$. We'll add it later.

$$\det\begin{pmatrix} \cos\theta-\lambda & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta-\lambda \end{pmatrix} = (\cos\theta-\lambda)(-\cos\theta-\lambda) - \sin^2\theta\, e^{i\varphi}e^{-i\varphi}$$

$= -\cos^2\theta + \lambda^2 - \sin^2\theta = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$ are the reduced eigenvalues

$$\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \cos\theta\, a + \sin\theta\, e^{-i\varphi}\, b = a$$

$$\Rightarrow b = \frac{1-\cos\theta}{\sin\theta}e^{i\varphi}a$$

PS9.EP.2

To normalize, we need $1 = |a|^2 + |b|^2 = |a|^2\left[1 + \frac{(1-\cos\theta)^2}{\sin^2\theta}\right] = \frac{\sin^2\theta + 1 - 2\cos\theta + \cos^2\theta}{\sin^2\theta}|a|^2$

$= \frac{2(1-\cos\theta)}{\sin^2\theta}|a|^2 \Rightarrow a = \frac{\sin\theta}{\sqrt{2(1-\cos\theta)}}$

We find $\begin{pmatrix} \frac{\sin\theta}{\sqrt{2(1-\cos\theta)}} \\ \sqrt{\frac{1-\cos\theta}{2}}e^{i\varphi} \end{pmatrix} = |J_n + \tfrac{1}{2}\rangle$ is an eigenvector of J_n with eigenvalue $+\frac{\hbar}{2}$

Meanwhile,

$$\begin{pmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -a \\ -b \end{pmatrix} \Rightarrow \cos\theta\, a + \sin\theta e^{-i\varphi} b = -a$$

$\Rightarrow a = \frac{-\sin\theta e^{-i\varphi}}{1+\cos\theta} b$. To normalize, $1 = |a|^2 + |b|^2$

$= |b|^2\left[1 + \frac{\sin^2\theta}{(1+\cos\theta)^2}\right] = |b|^2\left(\frac{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}{(1+\cos\theta)^2}\right) = |b|^2 \frac{2\cancel{(1+\cos\theta)}}{(1+\cos\theta)^{\cancel{2}}}$

$\Rightarrow b = \sqrt{\frac{1+\cos\theta}{2}}$

We find $\begin{pmatrix} \frac{-\sin\theta e^{-i\varphi}}{\sqrt{2(1+\cos\theta)}} \\ \sqrt{\frac{1+\cos\theta}{2}} \end{pmatrix} = |J_n - \tfrac{1}{2}\rangle$ is an eigenvector of J_n with eigenvalue $-\frac{\hbar}{2}$.

We will measure $+\frac{\hbar}{2}$ with probability $|\langle J_n + \tfrac{1}{2} | \tfrac{1}{2}\tfrac{1}{2}\rangle|^2 = \frac{\sin^2\theta}{2(1-\cos\theta)}$

We will measure $-\frac{\hbar}{2}$ with probability $|\langle J_n - \tfrac{1}{2} | \tfrac{1}{2}\tfrac{1}{2}\rangle|^2 = \frac{\sin^2\theta}{2(1+\cos\theta)}$

Thus, $\langle J_n\rangle = \frac{\hbar}{2}\left[\frac{\sin^2\theta}{2(1-\cos\theta)} - \frac{\sin^2\theta}{2(1+\cos\theta)}\right] = \frac{\hbar\sin^2\theta}{4}\left(\frac{1}{1-\cos\theta} - \frac{1}{1+\cos\theta}\right)$

$= \frac{\hbar\sin^2\theta}{4}\left(\frac{1+\cos\theta - 1 + \cos\theta}{1-\cos^2\theta}\right) = \frac{\hbar\cancel{\sin^2\theta}}{{}_2\cancel{4}}\frac{\cancel{2}\cos\theta}{\cancel{\sin^2\theta}} = \frac{\hbar}{2}\cos\theta$

PS9.EP2.3

(e) The smallest value we can get for J_n is $-\frac{\hbar}{2}$. When we get $-\frac{\hbar}{2}$, then the system must be in the state $|J_n -\frac{1}{2}\rangle = \begin{pmatrix} \frac{-\sin\theta\, e^{-i\varphi}}{\sqrt{2(1+\cos\theta)}} \\ \sqrt{\frac{1+\cos\theta}{2}} \end{pmatrix}$

When we then measure J_z, we can get:

$+\frac{\hbar}{2}$ with amplitude $\frac{-\sin\theta\, e^{-i\varphi}}{\sqrt{2(1+\cos\theta)}} \Rightarrow$ Probability $= \frac{\sin^2\theta}{2(1+\cos\theta)}$

$-\frac{\hbar}{2}$ with amplitude $\sqrt{\frac{1+\cos\theta}{2}} \Rightarrow$ Probability $= \frac{1+\cos\theta}{2}$

Side note: This implies $\langle J_z\rangle = \frac{\hbar}{2}\left(\frac{\sin^2\theta}{2(1+\cos\theta)} - \frac{1+\cos\theta}{2}\right)$

$= \frac{\hbar}{4}\left[\frac{\sin^2\theta - (1+\cos\theta)^2}{1+\cos\theta}\right] = \frac{\hbar}{4}\left[\frac{\sin^2\theta - 1 - 2\cos\theta - \cos^2\theta}{1+\cos\theta}\right] = \frac{\hbar}{4}\left(\frac{-2\cos\theta - 2\cos^2\theta}{1+\cos\theta}\right)$

$= -\frac{\hbar}{2}\cos\theta$, which is what part (c) would lead us to expect. ✓

Additional comment: In the above construction of $|J_n +\frac{1}{2}\rangle$ and $|J_n -\frac{1}{2}\rangle$, I didn't worry about their relative phases. Thus, while we know ~~[illegible]~~ $J_{n\pm}|J_n \mp\frac{1}{2}\rangle = C_\pm |J_n \pm\frac{1}{2}\rangle$ with $|C_\pm| = 1$, there is no guarantee that $C_\pm = +1$.