Physics 606 -- Problem Set 8

Due Monday, April 20, 2009 (drop it off at the Cyclotron any time)

Do the following problems from Merzbacher, Chapter 10:

Exercises 10.8, 10.9, 10.12, 10.14, 10.16, and 10.25
End-of-chapter problems 2 and 3

Also do the following variation on Exercise 10.18:

If I haven't made a mistake in my reasoning somewhere, then the only "eigenvector" of the raising operator $a^\dagger$ is the null vector $\mathbf{0}$. Prove me right, or prove me wrong.

Note: Typically, we don't consider $\mathbf{0}$ a valid eigenvector, since $A\mathbf{0} = A'\mathbf{0}$, for any linear operator A and any eigenvalue A'.

Chapter 10, Exercise 10.8

The ground state of the harmonic oscillator is:

$$\psi_0(x) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} \exp\left(-\frac{1}{2}\frac{m\omega}{\hbar}x^2\right)$$

In 10.66, if we set $\langle x\rangle = 0$, $\langle p_x\rangle = 0$, we find:

Eq 10.66(mod): $\psi(x) = \frac{1}{(2\pi(\Delta x)^2)^{1/4}} \exp\left[-\frac{1}{2}\frac{x^2}{2(\Delta x)^2}\right]$

These two expressions are equivalent when $2(\Delta x)^2 = \frac{\hbar}{m\omega}$

$$\Rightarrow \boxed{\Delta x = \sqrt{\frac{\hbar}{2m\omega}}}$$

Chapter 10, Exercise 10.9

The kinetic energy operator of the harmonic oscillator is

$T = \frac{p^2}{2m}$. The eigenfns are $\frac{1}{\sqrt{2\pi\hbar}} e^{i\frac{p_x x}{\hbar}}$ with continuous eigenvalues $\frac{p_x^2}{2m}$ for all values of p_x.

The potential energy operator of the harmonic oscillator is

$V(x) = \frac{1}{2} m\omega^2 x^2$. The eigenfns are $\delta(x - x_0)$ with continuous eigenvalues $\frac{1}{2} m\omega^2 x_0^2$ for all values of x_0.

$H = T + V$, but T + V don't commute. Thus, we can't diagonize them simultaneously. We can't expect their eigenvalues will have an simple relationship to those of H.

Ex10.12.1

Chapter 10, Exercise 10.12

Eq. 10.77: $a|0\rangle = 0 \Rightarrow \left(x + i\frac{p}{m\omega}\right)\psi_0(x) = \left(x + \frac{\hbar}{m\omega}\frac{d}{dx}\right)\psi_0(x) = 0$

$$\Rightarrow \frac{d\psi_0}{dx} = -\frac{m\omega}{\hbar}x\,\psi_0(x) \Rightarrow \psi_0(x) \propto \exp\left\{-\frac{1}{2}\frac{m\omega}{\hbar}x^2\right\}$$

Pf: $\psi_0 = e^{f(x)} \Rightarrow \frac{d\psi_0}{dx} = e^{f(x)}f'(x) = -\frac{m\omega}{\hbar}x\,e^{f(x)} \Rightarrow f'(x) = -\frac{m\omega}{\hbar}x \Rightarrow$

$$f(x) = -\frac{1}{2}\frac{m\omega}{\hbar}x^2 + \text{const.}$$

The proportionality constant is chosen so that $\int_{-\infty}^{+\infty}|\psi_0|^2dx = 1$:

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left\{-\frac{1}{2}\frac{m\omega}{\hbar}x^2\right\}.$$

Eq. 10.88 gives $|n\rangle = \frac{1}{\sqrt{n}}a^+|n-1\rangle$. We've just seen that $|0\rangle$ works.

Assume $|n-1\rangle = \psi_{n-1}$. Then

$$|n\rangle = \frac{1}{\sqrt{n}}a^+|n-1\rangle = \frac{1}{\sqrt{n}}\sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{i}{m\omega}\frac{\hbar}{i}\frac{d}{dx}\right)\psi_{n-1}(x)$$

$$= \frac{1}{\sqrt{n}}\left(\sqrt{\frac{m\omega}{2\hbar}}x - \sqrt{\frac{\hbar}{2m\omega}}\frac{d}{dx}\right)\frac{1}{\sqrt{2^{n-1}(n-1)!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left\{\frac{-1}{2}\frac{m\omega}{\hbar}x^2\right\}H_{n-1}\left(\sqrt{\frac{m\omega}{\hbar}}x\right)$$

$$= \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\sqrt{\frac{m\omega}{\hbar}}x - \sqrt{\frac{\hbar}{m\omega}}\frac{d}{dx}\right)\exp\left\{-\frac{1}{2}\frac{m\omega}{\hbar}x^2\right\}H_{n-1}\left(\sqrt{\frac{m\omega}{\hbar}}x\right)$$

$$= \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\xi - \frac{d}{d\xi}\right)\left(e^{-\frac{1}{2}\xi^2}H_{n-1}(\xi)\right)$$

$$= \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left[\xi e^{-\frac{1}{2}\xi^2}H_{n-1}(\xi) + \xi e^{-\frac{1}{2}\xi^2}H_{n-1}(\xi) - e^{-\frac{1}{2}\xi^2}\frac{dH_{n-1}(\xi)}{d\xi}\right]$$

$$= \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-\frac{1}{2}\xi^2}\left[2\xi H_{n-1}(\xi) - \frac{dH_{n-1}(\xi)}{d\xi}\right]$$

Ex10.12.2

In Problem Set 5, we derived:

$$\frac{dH_n}{d\xi} = 2nH_{n-1} \quad \text{and} \quad H_{n+1} - 2\xi H_n + 2n H_{n-1} = 0$$

$$\Rightarrow H_{n+1} = 2\xi H_n - \frac{dH_n}{d\xi} \Rightarrow H_n = 2\xi H_{n-1} - \frac{dH_{n-1}}{d\xi}, \text{ and}$$

$$|n\rangle = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{1}{2}\xi^2} H_n(\xi)$$, consistent with Eq. 5.39 and Eq. 10.96.

Chapter 10, Exercise 10.14

$R_\lambda^+ a R_\lambda = e^{i\lambda} a \Rightarrow a R_\lambda = e^{i\lambda} R_\lambda a$.

Then $a R_\lambda |\alpha\rangle = e^{i\lambda} R_\lambda a |\alpha\rangle = e^{i\lambda} R_\lambda \alpha |\alpha\rangle = \alpha e^{i\lambda} R_\lambda |\alpha\rangle$

$\Rightarrow R_\lambda |\alpha\rangle$ is an eigenvector of a with eigenvalue $\alpha e^{i\lambda}$.

Thus, $R_\lambda |\alpha\rangle = c' |e^{i\lambda}\alpha\rangle$, where c' is some constant.

But $\langle R_\lambda \alpha | R_\lambda \alpha \rangle = |c'|^2 \langle e^{i\lambda}\alpha | e^{i\lambda}\alpha \rangle = \langle \alpha | R_\lambda^+ R_\lambda | \alpha \rangle$

$= \langle \alpha | \alpha \rangle = 1$, since R_λ is unitary. Thus, $|c'| = 1$ and

$R_\lambda |\alpha\rangle = c' |e^{i\lambda}\alpha\rangle$, with c' a phase factor. ✓

R_λ serves to "rotate" all of the eigenvalues and eigenvectors of a by an angle λ about the origin of the complex plane, as shown in Figure 10.1.

Chapter 10, Exercise 10.16

$$a|\alpha\rangle = \alpha|\alpha\rangle \Rightarrow a\sum_n |n\rangle\langle n|\alpha\rangle = \alpha\sum_m |m\rangle\langle m|\alpha\rangle$$

$a|n\rangle = \sqrt{n}\,|n-1\rangle$, by Eq. 10.87a, so we have:

$$\sum_n \sqrt{n}\,|n-1\rangle\langle n|\alpha\rangle = \sum_m \alpha|m\rangle\langle m|\alpha\rangle$$

Take $\langle i|$ on both sides. Then:

$$\sqrt{i+1}\,\langle i+1|\alpha\rangle = \alpha\langle i|\alpha\rangle \Rightarrow \boxed{\langle i|\alpha\rangle = \frac{\alpha}{\sqrt{i}}\langle i-1|\alpha\rangle}$$

Iterating, we find $\langle n|\alpha\rangle = \frac{\alpha^n}{\sqrt{n!}}\langle 0|\alpha\rangle$, so that

$$|\alpha\rangle = \langle 0|\alpha\rangle \sum_n |n\rangle \frac{\alpha^n}{\sqrt{n!}}.$$

But $1 = \langle\alpha|\alpha\rangle = |\langle 0|\alpha\rangle|^2 \sum_n \frac{(|\alpha|^2)^n}{n!} = |\langle 0|\alpha\rangle|^2 e^{|\alpha|^2}$

$\Rightarrow |\langle 0|\alpha\rangle|^2 = e^{-|\alpha|^2}$, and we can choose $\langle 0|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}$

Substituting, we find $|\alpha\rangle = \sum_n |n\rangle e^{-\frac{1}{2}|\alpha|^2}\frac{\alpha^n}{\sqrt{n!}}$, Eq. 10.109.

Chapter 10, Variation on 10.18:

Assume $a^\dagger|v\rangle = v|v\rangle$, for some vector v.

Let $|v\rangle = \sum_i c_i |i\rangle$.

Assume $c_i = 0$ for $i<j$, $c_j \neq 0$. Then: $|v\rangle = \sum_{i \geq j} c_i |i\rangle$.

$$a^\dagger|v\rangle = \sum_{i=j}^{\infty} c_i a^\dagger |i\rangle = \sum_{i=j}^{\infty} c_i \sqrt{i+1}\,|i+1\rangle = \sum_{k=j}^{\infty} v c_k |k\rangle$$

Take $\langle j|$ on both sides.

On the left-hand side, $i+1$ is never equal to j, so we get 0.

On the right-hand side, we get $vc_j \Rightarrow vc_j = 0$.

c_j was assumed $\neq 0$, so we must have $\boxed{v=0}$

Now take $\langle j+1|$ on both sides. We find $c_j\sqrt{j+1} = 0 \cdot c_{j+1} = 0$

$\Rightarrow c_j = 0$.

This contradicts our original assumption that we could choose $c_j \neq 0$.

Thus, we must have <u>all</u> $c_n = 0$, and only $\vec{0}$ is an "eigenvector".

Chapter 10, Exercise 10.25

For $[a,a^\dagger]=1$, $x=\sqrt{\frac{\hbar}{2m\omega}}\left(a+a^\dagger\right)$,

$$p_x=\sqrt{\frac{m\hbar\omega}{2}}\, i\left(a^\dagger-a\right),$$

we found $(\Delta x)^2=\frac{\hbar}{2m\omega}$, $(\Delta p_x)^2=\frac{\hbar m\omega}{2}$ when $a|\alpha\rangle=\alpha|\alpha\rangle$

Here $[b,b^\dagger]=1$, $x=\sqrt{\frac{\hbar}{2m\omega'}}\left(b+b^\dagger\right)$

$$p_x=\sqrt{\frac{m\hbar\omega'}{2}}\, i\left(b^\dagger-b\right),$$

so the same derivation shows $(\Delta x')^2=\frac{\hbar}{2m\omega'}$, $(\Delta p_x')^2=\frac{\hbar m\omega'}{2}$ when $b|\beta\rangle=\beta|\beta\rangle$

Eqs. 10.126 + 10.127 define:

$$\lambda=\frac{\left(\sqrt{\frac{\omega'}{\omega}}+\sqrt{\frac{\omega}{\omega'}}\right)}{2}, \quad \nu=\frac{\sqrt{\frac{\omega'}{\omega}}-\sqrt{\frac{\omega}{\omega'}}}{2}$$

Then $\lambda-\nu=\frac{2\sqrt{\frac{\omega}{\omega'}}}{2}=\sqrt{\frac{\omega}{\omega'}}$, $\lambda+\nu=\sqrt{\frac{\omega'}{\omega}}$, and we have:

$$\left.\begin{aligned}(\Delta x')^2&=\frac{\hbar}{2m\omega'}=\frac{\hbar}{2m\omega}\frac{\omega}{\omega'}=(\Delta x)^2(\lambda-\nu)^2\\(\Delta p_x')^2&=\frac{\hbar m\omega'}{2}=\frac{\hbar m\omega}{2}\frac{\omega'}{\omega}=(\Delta p_x)^2(\lambda+\nu)^2\end{aligned}\right\}\Rightarrow \text{Eq. 10.131.} \checkmark$$

Chapter 10, Problem 2

For convenience, let's take $0<x<2a$, so $\psi_n(x) = \begin{cases} \frac{1}{\sqrt{a}} \sin\left(\frac{n\pi x}{2a}\right) & 0<x<2a \\ 0 & \text{otherwise} \end{cases}$

By symmetry $\langle x \rangle = a$ and $\langle p_x \rangle = 0$

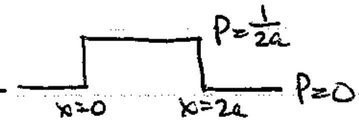
$$\langle x^2 \rangle = \frac{1}{a}\int_0^{2a} \sin\left(\frac{n\pi x}{2a}\right) x^2 \sin\left(\frac{n\pi x}{2a}\right) dx$$

$$\int x^2 \sin^2 x\, dx = \frac{x^3}{6} - \left(\frac{x^2}{4} - \frac{1}{8}\right)\sin 2x - \frac{x\cos 2x}{4} \quad \text{(my integral tables)}$$

Thus, $y = \frac{n\pi x}{2a} \Rightarrow \langle x^2 \rangle = \frac{1}{a}\int_0^{n\pi} \left(\frac{2a}{n\pi}\right)^2 y^2 \sin^2 y \left(\frac{2a}{n\pi}\right) dy = \frac{8a^2}{n^3\pi^3}\int_0^{n\pi} y^2 \sin^2 y\, dy$

$$= \frac{8a^2}{n^3\pi^3}\left\{\frac{(n\pi)^3}{6} - \frac{n\pi}{4}\right\} = a^2\left(\frac{4}{3} - \frac{2}{(n\pi)^2}\right)$$

$$\Rightarrow (\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2 = \boxed{a^2\left(\frac{1}{3} - \frac{2}{(n\pi)^2}\right)}$$

In the large-n limit, $\Delta x = \frac{a}{\sqrt{3}}$, which is the RMS of 

$$\langle p_x^2 \rangle = \langle \psi_n | p_x^2 | \psi_n \rangle = \langle p_x \psi_n | p_x \psi_n \rangle$$

$$= \frac{\hbar^2}{a}\int_0^{2a} \frac{n\pi}{2a}\cos\left(\frac{n\pi x}{2a}\right) \frac{n\pi}{2a}\cos\left(\frac{n\pi x}{2a}\right) dx = \frac{\hbar^2}{a}\left(\frac{n\pi}{2a}\right)^2 \int_0^{2a} \cos^2\left(\frac{n\pi x}{2a}\right) dx$$

$$\boxed{= \left(\frac{\hbar n\pi}{2a}\right)^2}$$

Thus, $\Delta x \Delta p_x = a\sqrt{\frac{1}{3} - \frac{2}{(n\pi)^2}}\ \frac{\hbar \pi n}{2a} = \boxed{\frac{\hbar}{2}\sqrt{\frac{(n\pi)^2}{3} - 2}\ , n=1,2,3,\ldots}$

The minimum occurs when $n=1$ and $\Delta x \Delta p_x \sim 1.14\,\frac{\hbar}{2}$.
The uncertainty principle is always satisfied.

Chapter 10, Problem 3

By parity symmetry, $\langle n|x|n\rangle = 0$, $\langle n|p_x|n\rangle = 0$

$$\langle x^2\rangle = \langle n|x^2|n\rangle = \sum_i \langle n|x|i\rangle\langle i|x|n\rangle = \langle n|x|n+1\rangle\langle n+1|x|n\rangle$$
$$+\langle n|x|n-1\rangle\langle n-1|x|n\rangle$$
$$= \frac{\hbar}{2m\omega}\left\{\langle n|a|n+1\rangle\langle n+1|a^+|n\rangle + \langle n|a^+|n-1\rangle\langle n-1|a|n\rangle\right\}$$

$$= \frac{\hbar}{2m\omega}\left\{(n+1)+n\right\} = (2n+1)\frac{\hbar}{2m\omega} = (\Delta x)^2$$

$$\langle p_x^2\rangle = \langle n|p_x^2|n\rangle = \langle n|\left(i\sqrt{\frac{m\hbar\omega}{2}}(a^+-a)\right)^2|n\rangle$$

$$= -\frac{m\hbar\omega}{2}\langle n|(a^+-a)^2|n\rangle = -\frac{m\hbar\omega}{2}\langle n|-a^+a-aa^+|n\rangle$$ (The a^2 and a^{+2} terms give zero.)

$$= \frac{m\hbar\omega}{2}\left\{\langle n|a^+|n-1\rangle\langle n-1|a|n\rangle + \langle n|a|n+1\rangle\langle n+1|a^+|n\rangle\right\}$$

$$= \frac{m\hbar\omega}{2}\left\{n+n+1\right\} = (2n+1)\frac{m\hbar\omega}{2} = (\Delta p_x)^2$$

$$\Rightarrow (\Delta x)^2(\Delta p_x)^2 = \frac{\hbar}{2\cancel{m\omega}}(2n+1)\frac{\hbar\cancel{m\omega}}{2}(2n+1) = \left[\frac{\hbar}{2}(2n+1)\right]^2$$

$$\Rightarrow \boxed{\Delta x\,\Delta p_x = \frac{\hbar}{2}(2n+1)}$$