Physics 606 -- Problem Set 7

Due Thursday, April 9, 2009

Do the following problems from Merzbacher, Chapter 9:

Exercises 9.8, 9.13, 9.15, 9.16, 9.17, 9.20
End-of-chapter problem 1

Also do the following problem from Chapter 10:

Exercises 10.2, 10.5

Chapter 9, Exercise 9.8

Let $\Psi_a = \sum a_i \Psi_i$ and $\Psi_b = \sum b_i \Psi_i$, where Ψ_i is a complete, orthonormal basis. Then

$$F_A(\Psi_a, \Psi_b) = F_A\left(\sum_i a_i \Psi_i, \sum_j b_j \Psi_j\right) = \sum_{i,j} a_i^* F_A(\Psi_i, \Psi_j)\, b_j$$

Define $A_{ij} = F_A(\Psi_i, \Psi_j)$. Then $F_A(\Psi_a, \Psi_b) = \sum_{i,j} a_i^* A_{ij} b_j$

$$= (\Psi_a, A\Psi_b).$$

Chapter 9, Exercise 9.13

$$\langle (AB)^\dagger \Psi_a | \Psi_b \rangle = \langle \Psi_a | AB \Psi_b \rangle = \langle A^\dagger \Psi_a | B \Psi_b \rangle = \langle B^\dagger A^\dagger \Psi_a | \Psi_b \rangle$$

$\Rightarrow (AB)^\dagger = B^\dagger A^\dagger$. (i.e., Eq. 9.67 "proof left to reader")

Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be unitary. Then $\mathcal{U}_1\mathcal{U}_2(\mathcal{U}_1\mathcal{U}_2)^\dagger = \mathcal{U}_1(\mathcal{U}_2\mathcal{U}_2^\dagger)\mathcal{U}_1^\dagger$

$= \mathcal{U}_1\mathcal{U}_1^\dagger = I$ and $(\mathcal{U}_1\mathcal{U}_2)^\dagger\mathcal{U}_1\mathcal{U}_2 = \mathcal{U}_2^\dagger(\mathcal{U}_1^\dagger\mathcal{U}_1)\mathcal{U}_2 = \mathcal{U}_2^\dagger\mathcal{U}_2 = I$

$\Rightarrow \mathcal{U}_1\mathcal{U}_2$ is unitary.

Chapter 9, Exercise 9.15

$$\sum_i |\langle a|A|K_i\rangle|^2 = \sum_i \langle a|A|K_i\rangle\langle a|A|K_i\rangle^* \overset{Eq.\,9.96}{=} \sum_i \langle a|A|K_i\rangle\langle K_i|A^+|a\rangle$$

But $\sum_i |K_i\rangle\langle K_i| = I \Rightarrow \quad \sum_i |\langle a|A|K_i\rangle|^2 = \langle a|AA^+|a\rangle$

If A is unitary, $AA^+ = I$, and $\sum_i |\langle a|A|K_i\rangle|^2 = 1$

Chapter 9, Exercise 9.16

Consider $cA^iB^jC^k$:

$$(S^{-1}AS)^i = S^{-1}ASS^{-1}ASS^{-1}AS \cdots S^{-1}AS.$$

All intermediate $SS^{-1} = I \Rightarrow (S^{-1}AS)^i = S^{-1}A^iS$.

Then $c(S^{-1}AS)^i(S^{-1}BS)^j(S^{-1}CS)^k$

$$= c\,S^{-1}A^i\overbrace{SS^{-1}}^{I}B^j\overbrace{SS^{-1}}^{I}C^kS = c\,S^{-1}A^iB^jC^kS = S^{-1}(cA^iB^jC^k)S$$

This applies to all monomial expressions, independent of the order of the A's, B's, C's, their commutation, or whatever. This will then also apply to linear combinations.

What about negative powers?

$$\text{Consider } \frac{1}{AB-C} \rightarrow \frac{1}{S^{-1}ASS^{-1}BS - S^{-1}CS} = \frac{1}{S^{-1}(AB-C)S} =$$

$$\left[S^{-1}(AB-C)S\right]^{-1} = S^{-1}(AB-C)^{-1}S = S^{-1}\frac{1}{AB-C}S \Rightarrow \text{it works!}$$

Thus, this applies for all algebraic expressions of A, B, C and constant numbers.

(Would only work when c is a constant matrix not αI when $[c,S] = [c,S^{-1}] = 0$)

Ex 9.17.1

Chapter 9, Exercise 9.17

$(UHU^\dagger)^\dagger = U^{\dagger\dagger}H^\dagger U^\dagger = UHU^\dagger$ when H is hermitian $\Rightarrow$
transformed hermitian operators are hermitian.

Let U be a unitary operator and S (unitary) provide an active transformation $U \to \bar{U} = SUS^\dagger \Rightarrow$

$$\bar{U}\bar{U}^\dagger = (SUS^\dagger)(SUS^\dagger)^\dagger = SU\overbrace{S^\dagger S}^{I}U^\dagger S^\dagger = S\overbrace{UU^\dagger}^{I}S^\dagger = SS^\dagger = I$$
$$\bar{U}^\dagger\bar{U} = (SUS^\dagger)^\dagger(SUS^\dagger) = SU^\dagger S^\dagger SUS^\dagger = SU^\dagger US^\dagger = I$$
$\Rightarrow \bar{U}$ is unitary.

Let A be a normal operator. Then:

$$[\bar{A}, \bar{A}^\dagger] = \bar{A}\bar{A}^\dagger - \bar{A}^\dagger\bar{A} = UAU^\dagger(UAU^\dagger)^\dagger - (UAU^\dagger)^\dagger(UAU^\dagger)$$
$$= UA\overbrace{U^\dagger U}^{I}A^\dagger U^\dagger - UA^\dagger\overbrace{U^\dagger U}^{I}AU^\dagger = U(AA^\dagger - A^\dagger A)U^\dagger$$
$$= U[A, A^\dagger]U^\dagger = 0,$$ so $\bar{A}$ is also normal.

A diagonalized hermitian matrix is symmetric. When not diagonal, it must still be hermitian, but not necessarily symmetric. Example:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix} \Rightarrow U^\dagger = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}. \quad UU^\dagger = U^\dagger U = I$$

$$UHU^\dagger = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{-i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{pmatrix}$$

which is hermitian, but not symmetric.

Ex9.17.2

Side note: I actually came to my counterexample for symmetric matrices in reverse logic from that above. My reasoning was:

Let $A = A^t$. Then $(UAU^+)^t = (U^+)^t A^t U^t$

$= U^* A (U^*)^+$. Thus, A is not the key issue here, and can be chosen for simple calculations (e.g., $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$). But we must focus on cases where $U^* \neq U$. There are "lots" of those, and the first one I tried worked fine. (See above)

Chapter 9, Exercise 9.20

The derivation of 9.130 was:

$$\text{trace } AB = \sum_i \langle K_i | AB | K_i \rangle = \sum_i \langle K_i | A \Big(\sum_j |L_j\rangle\langle L_j| \Big) B | K_i \rangle$$

$$= \sum_{i,j} \langle K_i | A | L_j \rangle \langle L_j | B | K_i \rangle = \sum_{i,j} \langle L_j | B | K_i \rangle \langle K_i | A | L_j \rangle$$

$$= \sum_j \langle L_j | B \Big(\sum_i |K_i\rangle\langle K_i| \Big) A | L_j \rangle = \sum_j \langle L_j | BA | L_j \rangle$$

$$= \text{trace } BA.$$

This derivation always works for finite matrices.

For infinite dimensional matrices, one needs the sums over i and j to converge sufficiently rapidly so that $\sum_i \Big(\sum_j \Big) = \sum_j \Big(\sum_i \Big)$.

For $xp - px = i\hbar I$, the trace on the right-hand side is infinite, so there is no guarantee that the interchange of sums on the left-hand side is valid.

Chapter 9, Problem 1

By the closure relation (Eq. 4.42), $\delta^{(3)}(\vec{r}-\vec{r}') = \sum_i \psi_i(\vec{r})\,\psi_i^*(\vec{r}')$

If a particle is known to be at position $\vec{r}$, then:

$$\psi(\vec{r}') = \delta^{(3)}(\vec{r}'-\vec{r}) = \sum_i \psi_i(\vec{r}')\,\psi_i^*(\vec{r})$$

Then $c_n = \langle \psi_n | \psi \rangle = \sum_i \langle \psi_n | \psi_i \rangle\, \psi_i^*(\vec{r}) = \sum_i \delta_{ni}\, \psi_i^*(\vec{r}) = \psi_n^*(\vec{r})$

The probability to measure E_n is $|c_n|^2 = |\psi_n(\vec{r})|^2$.

Chapter 10, Exercise 10.2

Let A be an operator where $A|A_i\rangle = A_i'|A_i\rangle$ and $|A_i\rangle$ form a complete, orthonormal basis $\Rightarrow \sum_i |A_i\rangle\langle A_i| = I$. Then:

$$A = AI = \sum_i A|A_i\rangle\langle A_i| = \sum_i A_i'|A_i\rangle\langle A_i| = \sum_i |A_i\rangle A_i'\langle A_i|,$$

so $A = \sum_i |A_i\rangle A_i'\langle A_i|$.

Then $A^{+} = \sum_i \left(|A_i\rangle A_i'\langle A_i|\right)^{+} = \sum_i |A_i\rangle A_i'^{*}\langle A_i|$, and

$$AA^{+} = \sum_i \left(|A_i\rangle A_i'\langle A_i|\right)\left(\sum_j |A_j\rangle A_j'^{*}\langle A_j|\right) = \sum_{i,j} |A_i\rangle A_i'\langle A_i|A_j\rangle A_j'^{*}\langle A_j|$$

but $\langle A_i|A_j\rangle = \delta_{ij} \Rightarrow \; = \sum_i |A_i\rangle A_i' A_i'^{*}\langle A_i|$

Likewise, $A^{+}A = \sum_i |A_i\rangle A_i'^{*} A_i'\langle A_i|$, and $[A, A^{+}] = 0$. ✓

Chapter 10, Exercise 10.5

A normal n-dimensional matrix A can be written as:

$$A = \sum_i |A_i\rangle A_i' \langle A_i| ,$$ where A_i' are the eigenvalues and satisfy the characteristic equation, and $|A_i\rangle$ are the corresponding eigenvectors which are complete and can be chosen to ~~to~~ be orthonormal.

Let ~~P(λ) be the P(λ)=0 be~~ $P(\lambda) = \det(A - \lambda I) = 0$ be the characteristic equation.

$$\langle A_i | A_j \rangle = \delta_{ij} \Rightarrow A^m = \sum_i |A_i\rangle A_i'^m \langle A_i| ,$$ so:

$$P(A) = \sum_i |A_i\rangle P(A_i') \langle A_i| .$$

But each $P(A_i')$ satisfies the characteristic eq $\Rightarrow P(A_i') = 0 \Rightarrow$

$$P(A) = \sum_i |A_i\rangle \, 0 \, \langle A_i| = 0 ,$$ and A satisfies $P(A) = 0$.

P is a polynomial of degree n of the form $(-1)^n \lambda^n + p_{n-1} \lambda^{n-1} + \cdots p_0 = 0$

$\Rightarrow (-1)^n A^n = -p_{n-1} A^{n-1} - p_{n-2} A^{n-2} - \cdots - p_0 I$, and A^n can be expanded as a polynomial ~~of order~~ in A of order n−1 or less.