Physics 606 -- Problem Set 3

Due Thursday, February 19, 2009

Do the following problems from Merzbacher:

Chapter 4:	Exercises 4.11, 4.13, and 4.17
Chapter 4:	End-of-chapter Problem 3

Chapter 4, Exercise 11

If $\psi = c_1\psi_1 + c_2\psi_2$, then $\langle U\psi | U\psi \rangle = \langle \psi | \psi \rangle$,

$$\langle U\psi_1 | U\psi_1 \rangle = \langle \psi_1 | \psi_1 \rangle, \langle U\psi_2 | U\psi_2 \rangle = \langle \psi_2 | \psi_2 \rangle$$

$$\langle U\psi | U\psi \rangle = c_1^* c_1 \langle U\psi_1 | U\psi_1 \rangle + c_1^* c_2 \langle U\psi_1 | U\psi_2 \rangle$$

$$+ c_2^* c_1 \langle U\psi_2 | U\psi_1 \rangle + c_2^* c_2 \langle U\psi_2 | U\psi_2 \rangle$$

$$= c_1^* c_1 \langle \psi_1 | \psi_1 \rangle + c_1^* c_2 \langle \psi_1 | \psi_2 \rangle$$

$$+ c_2^* c_1 \langle \psi_2 | \psi_1 \rangle + c_2^* c_2 \langle \psi_2 | \psi_2 \rangle$$

$$\Rightarrow \cancel{c_1^* c_2 \left[\langle U\psi_1 | U\psi_2 \rangle - \langle \psi_1 | \psi_2 \rangle \right] = c_2^* c_1 \left[\langle \psi_2 | \psi_1 \rangle - \langle U\psi_2 | U\psi_1 \rangle \right]}$$

$$c_1^* c_2 \langle U\psi_1 | U\psi_2 \rangle + c_2^* c_1 \langle U\psi_2 | U\psi_1 \rangle = c_1^* c_2 \langle \psi_1 | \psi_2 \rangle + c_2^* c_1 \langle \psi_2 | \psi_1 \rangle$$

$$\Rightarrow c_1^* c_2 \langle U\psi_1 | U\psi_2 \rangle + c_2^* c_1 \langle U\psi_1 | U\psi_2 \rangle^* = c_1^* c_2 \langle \psi_1 | \psi_2 \rangle + c_2^* c_1 \langle \psi_1 | \psi_2 \rangle^*$$

Let $c_1 = c_2 = 1 \Rightarrow \mathrm{Re}\left[\langle U\psi_1 | U\psi_2 \rangle \right] = \mathrm{Re}\left[\langle \psi_1 | \psi_2 \rangle \right]$

Let $c_1 = 1, c_2 = -i \Rightarrow \mathrm{Im}\left[\langle U\psi_1 | U\psi_2 \rangle \right] = \mathrm{Im}\left[\langle \psi_1 | \psi_2 \rangle \right]$

$$\Rightarrow \langle U\psi_1 | U\psi_2 \rangle = \langle \psi_1 | \psi_2 \rangle, \text{ for all } \psi_1, \psi_2$$

Chapter 4, Exercise 13

$D_\xi g(x) = g(x-\xi)$, by definition.

Also, if $g(x)$ is an eigenfunction of D_ξ with eigenvalue $D_\xi' = e^{-ik'\xi}$,

then $D_\xi g(x) = e^{-ik'\xi} g(x) = g(x-\xi)$.

Let $g(x) = e^{ik'x} u(x)$. Then $e^{-ik'\xi} g(x) = e^{ik'(x-\xi)} u(x) = g(x-\xi)$

$$= e^{ik'(x-\xi)} u(x-\xi)$$

$\Rightarrow$ we must have $u(x) = u(x-\xi)$, and u must be periodic with period ξ.

Chapter 4, Exercise 17

$$i\hbar\frac{\partial\psi}{\partial t}=H\psi \Rightarrow i\hbar\,\psi^*\frac{\partial\psi}{\partial t}=\psi^* H\psi$$

$$-i\hbar\frac{\partial\psi^*}{\partial t}=(H\psi)^* \Rightarrow -i\hbar\left(\frac{\partial\psi^*}{\partial t}\right)\psi=(H\psi)^*\psi$$

$$\Rightarrow i\hbar\frac{\partial}{\partial t}(\psi^*\psi)=\psi^*H\psi-(H\psi)^*\psi$$

$$\Rightarrow \frac{\partial}{\partial t}(\psi^*\psi)=\frac{1}{i\hbar}\left[\psi^*H\psi-(H\psi)^*\psi\right]$$

<u>Note</u>: $H\psi=\frac{1}{2m}\left(p-\frac{q}{c}A\right)^2\psi+q\phi\psi=\frac{1}{2m}\left[\frac{\hbar}{i}\vec{\nabla}-\frac{q}{c}A\right]^2\psi+q\phi\psi$

$$=\left(-\frac{\hbar^2}{2m}\nabla^2-\frac{\hbar q}{2mci}(\vec{\nabla}\cdot\vec{A})-\frac{2\hbar q}{2mci}\vec{A}\cdot\vec{\nabla}+\frac{q^2}{2mc^2}A^2\right)\psi+q\phi\psi$$

$$=-\frac{\hbar^2}{2m}\left[\nabla^2+\frac{q}{\hbar ci}(\vec{\nabla}\cdot\vec{A})+\frac{2q}{\hbar ci}\vec{A}\cdot\vec{\nabla}-\frac{q^2}{\hbar^2c^2}A^2\right]\psi+q\phi\psi$$

Thus: $$\frac{\partial}{\partial t}(\psi^*\psi)=\frac{-\hbar}{2mi}\Big\{\psi^*\nabla^2\psi+\psi^*\frac{q}{\hbar ci}(\vec{\nabla}\cdot\vec{A})\psi+\psi^*\frac{2q}{\hbar ci}\vec{A}\cdot\vec{\nabla}\psi-\cancel{\psi^*\frac{q^2}{\hbar^2c^2}A^2\psi}$$
$$-(\nabla^2\psi^*)\psi+\frac{q}{\hbar ci}(\vec{\nabla}\cdot\vec{A})\psi^*\psi+\frac{2q}{\hbar ci}\left[(\vec{A}\cdot\vec{\nabla})\psi^*\right]\psi+\cancel{\frac{q^2}{\hbar^2c^2}A^2\psi^*\psi}\Big\}$$

Consider $\vec{j}=\frac{\hbar}{2mi}\left\{\psi^*\left(\vec{\nabla}-\frac{iq}{\hbar c}\vec{A}\right)\psi-\left[\left(\vec{\nabla}-\frac{iq}{\hbar c}\vec{A}\right)\psi\right]^*\psi\right\}$

Then $$\vec{\nabla}\cdot\vec{j}=\frac{\hbar}{2mi}\Big\{\cancel{(\vec{\nabla}\psi^*)\cdot\vec{\nabla}\psi}-\frac{iq}{\hbar c}\vec{A}\cdot(\vec{\nabla}\psi^*)\psi+\psi^*\nabla^2\psi-\psi^*\frac{iq}{\hbar c}(\vec{\nabla}\cdot\vec{A})\psi$$
$$-\psi^*\frac{iq}{\hbar c}\vec{A}\cdot\vec{\nabla}\psi-(\nabla^2\psi^*)\psi-\frac{iq}{\hbar c}(\vec{\nabla}\cdot\vec{A})\psi^*\psi-\frac{iq}{\hbar c}(\vec{A}\cdot\vec{\nabla}\psi^*)\psi$$
$$-\cancel{(\vec{\nabla}\psi^*)\cdot(\vec{\nabla}\psi)}-\frac{iq}{\hbar c}\psi^*\vec{A}\cdot\vec{\nabla}\psi\Big\}$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}(\psi^*\psi)+\vec{\nabla}\cdot\vec{j}=0 \text{ w/ } \vec{j}=\frac{\hbar}{2mi}\left\{\psi^*\left(\vec{\nabla}-\frac{iq}{\hbar c}\vec{A}\right)\psi-\left[\left(\vec{\nabla}-\frac{iq}{\hbar c}\vec{A}\right)\psi\right]^*\psi\right\}}$$

Chapter 4, Problem 3

a) $\vec{A} = \frac{\Phi}{2\pi}\frac{\hat{k}\times\vec{r}}{(\hat{k}\times\vec{r})^2} \Rightarrow \vec{A} = \frac{\Phi}{2\pi}\begin{pmatrix} -\frac{y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \\ 0 \end{pmatrix}$

Then $B_x = \partial_y A_z - \partial_z A_y = 0 - 0 = 0.$

Likewise, $B_y = 0.$

$$B_z = \partial_x A_y - \partial_y A_x = \frac{\Phi}{2\pi}\left[\frac{(x^2+y^2)-x(2x)}{(x^2+y^2)^2} - (-)\frac{(x^2+y^2)-y(2y)}{(x^2+y^2)^2}\right]$$

$$= \frac{\Phi}{2\pi}\left[\frac{(x^2+y^2)+(x^2+y^2)-2x^2-2y^2}{(x^2+y^2)^2}\right] = 0 \text{ if } (x,y) \neq (0,0)$$

But if you integrate over a circle of radius R centered on the z-axis, you get:

$\int B\,dA = \int \vec{\nabla}\times\vec{A}\,dA = \oint \vec{A}\cdot d\vec{l} = \frac{\Phi}{2\pi R}\int dl = \Phi$, so there must be a magnetic flux Φ along the z-axis.

b) $\vec{A} = A_\phi\hat{\varphi}$, where $A_\phi = \frac{\Phi}{2\pi\rho}$. Then we have

$$\frac{\left(\vec{p}-\frac{q}{c}\vec{A}\right)^2}{2m}\psi_E = E\psi_E \Rightarrow \frac{1}{2m}\left(\frac{\hbar}{i\rho}\frac{d}{d\phi} - \frac{q\Phi}{2\pi\rho c}\right)^2\psi_E = E\psi_E$$

We still need ψ_E periodic $\Rightarrow \psi_E = \frac{1}{\sqrt{2\pi}}e^{in\phi} \Rightarrow$

Same final spectrum when $\frac{q\Phi}{2\pi c} = n'\hbar$. Then $E_n \to E_{n-n'}$

$$\left(\frac{\hbar}{i\rho}\frac{d}{d\phi} - \frac{q\Phi}{2\pi\rho c}\right)\psi_E = \left(\frac{\hbar n}{\rho} - \frac{q\Phi}{2\pi\rho c}\right)\psi_E \Rightarrow$$

$$E_n = \frac{1}{2m}\left(\frac{\hbar n}{\rho} - \frac{q\Phi}{2\pi\rho c}\right)^2 = \frac{1}{2m}\left[\frac{\hbar^2 n^2}{\rho^2} - 2\frac{\hbar n}{\rho}\frac{q\Phi}{2\pi\rho c} + \frac{q^2\Phi^2}{4\pi^2\rho^2c^2}\right]$$

$$= \frac{\hbar^2 n^2}{2m\rho^2} - \frac{\hbar n q\Phi}{2m\pi\rho c} + \frac{q^2\Phi^2}{8\pi^2 m\rho^2 c^2}$$

Same if $n'\hbar = \frac{q\Phi}{2\pi c}$