Do the following three problems from Griffiths:

4.9, pg 145  
Note: By “radial equation”, this problem means Eq. 4.37.
4.38, pg 190
4.57, pg 197-98  
Comment: I’d not seen this before Griffiths. It’s cool!

Also do the following four additional problems. The first two build on Problem 4.9 above, while the third follows up on Problem 4.38 (with a lot less arithmetic mess than Griffiths’ alternative, Problem 4.39).

1) Now consider the bound states of the finite spherical well (Prob. 4.9) that have \( l \neq 0 \). For this case, it’s more direct to use Eq. 4.35 as your radial Schrödinger equation. For the (classically allowed) interior region \( r < a \) where \( E > -V_0 \), the general solution is \( \alpha j_l(\kappa r) + \beta n_l(\kappa r) \), where \( \kappa = \sqrt{\frac{2m|E|}{\hbar^2}} \).

(a) For arbitrary \( l \), find the interior linear combination that obeys our boundary condition at \( r = 0 \).
(b) For arbitrary \( l \), find the exterior linear combination that obeys our boundary condition at \( r \to \infty \).
(c) For arbitrary \( l \), find the continuity condition that connects your results from parts (a) and (b) at \( r = a \). In general, this represents a very complicated transcendental equation.

2) Now let’s find the zero-energy bound state solutions to the finite square well for \( l > 0 \).
(a) For this case, the Eq. 4.35 version of the radial wave equation for the exterior region, \( r > a \), has simple solutions. Find the linear combination that obeys our boundary condition at \( r \to \infty \).
(b) Use the continuity condition at \( r = a \) to show that a zero-energy bound state with \( l > 0 \) will occur when \( j_{l,1}(ka) = 0 \). (FYI: This can be used to determine the number of bound states with \( l \).)

3) The three-dimensional harmonic oscillator wave functions that you found in Prob. 4.38 are not simultaneous eigenfunctions of \( L^2 \) and \( L_z \). However, we know that it’s possible to construct eigenfunctions of \( H \) for the 3D harmonic oscillator that are also eigenfunctions of \( L^2 \), \( L_z \), and parity. Combine the Gaussian factors that appear in your Prob. 4.38 eigenfunctions into a function of \( r \) that is independent of \( \theta \) and \( \phi \). Then use the parities, degeneracies \( d(n) \), remaining powers of \( x \), \( y \), and \( z \) that appear in the Cartesian eigenfunctions, and the functional form of \( r^m Y_l^m \) to infer the allowed values of \( l \) and \( m \) for the states of the harmonic oscillator with energy \( E_n \). Note: If you are doing a fancy calculation – or even thinking about a differential equation – you are on the wrong track!

4) The anti-commutator of two operators \( F \) and \( G \) is given by \( \{F,G\} = FG + GF \). Consider a physical system with a Hamiltonian \( H = \hbar \omega (C + \frac{1}{2}) \). Assume that its eigenvectors obey the eigenvalue condition \( C|c\rangle = c|c\rangle \), where the \( c \)'s are dimensionless, real-valued quantities. Furthermore, assume that \( C \) can be written as \( C = a\dagger a \), where \( a \) and \( a\dagger \) are an additional pair of operators that obey the anti-commutation relation \( \{a,a\dagger\} = 1 \).
(a) Find the anti-commutators \( \{C,a\dagger\} \) and \( \{C,a\} \).
(b) Determine \( a|c\rangle \) and \( a\dagger|c\rangle \).
(c) Now consider the additional condition \( a^2 = 0 \), in addition to everything given above. Show that \( C \) is a projection operator in this case.
(d) Determine all the possible values of the eigenvalue \( c \) for the special case in part (c).
Griffiths, Problem 4.9

For \( r < a \), Eq. 4.37 gives \(-\frac{k^2}{2m} \frac{d^2u}{dr^2} - V_0 u = Eu\)

\[ \Rightarrow \frac{d^2u}{dr^2} + \frac{2m}{k^2} (V_0 - 1/2) u = 0 \Rightarrow u = A \sin kr + B \cos kr.\]

We need \( u = 0 \) at the origin, so 
\[ u = \frac{A \sin kr}{\sqrt{2m/k^2 (V_0 - 1)/2}} \]

For \( r > a \), Eq. 4.37 gives \(-\frac{k^2}{2m} \frac{d^2u}{dr^2} = Eu\)

\[ \Rightarrow \frac{d^2u}{dr^2} - \frac{2m}{k^2} 1/2 E u = 0 \Rightarrow u = D e^{-kr} \text{ with } K = \sqrt{\frac{2m|E|}{k^2}} \]

At \( r = a \), we need: \( u \text{ cont } \Rightarrow A \sin ka = D e^{-ka} \)

\[ u' \text{ cont } \Rightarrow A k \cos ka = -K D e^{-ka} \]

\[ \Rightarrow k \cot ka = -K \text{ or } ka \cot ka = -Ka \]

\[ \text{with } (ka)^2 + (Ka)^2 = \frac{2mV_0 a^2}{k^2} \]

are the eigenvalue conditions.

\(-Ka\) is negative, so we need \( \cot ka \leq 0 \Rightarrow ka \geq \frac{\pi}{2} \)

Thus, there will be no bound state unless \( \frac{2mV_0 a^2}{k^2} \geq \left( \frac{\pi}{2} \right)^2 \)

\[ \Rightarrow V_0 a^2 > \frac{k^2 \pi^2}{2m} \]

as desired. (Griffiths has < because he says no state "if", whereas I said no state "unless".)
Griffiths, Problem 4.38

(a) \[-\frac{k^2}{2m} \nabla^2 \phi + \frac{1}{2} mc^2 \phi^2 \phi = E \phi \Rightarrow \]
\[-\frac{k^2}{2m} \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \right) \phi + \frac{1}{2} mc^2 \left( x^2 + y^2 + z^2 \right) \phi = E \phi \]

If \( \phi = \phi_x(x) \phi_y(y) \phi_z(z) \), then:
\[-\frac{k^2}{2m} \frac{\partial^2 \phi_x}{\partial x^2} + \frac{1}{2} mc^2 x^2 \phi_x = E_x \phi_x \]
with likewise for y and z
and \( E = E_x + E_y + E_z \).

The decoupled equation is just the 1-d harmonic oscillator,
with \( E_x = \hbar \omega \left( n_x + \frac{1}{2} \right) \).

Thus, \( E = E_x + E_y + E_z = \hbar \omega \left[ n_x + \frac{1}{2} + n_y + \frac{1}{2} + n_z + \frac{1}{2} \right] \Rightarrow \)
\( E = \hbar \omega \left( n + \frac{3}{2} \right) \) with \( n = n_x + n_y + n_z \).

(b) \( n_x \) can take any value from 0 to \( n \), for \( n+1 \) possibilities.
\( n_y \) can take any value from 0 to \( n - n_x \), for \( n - n_x + 1 \) possibilities.
\( n_z \) is then fixed to give \( n_x + n_y + n_z = n \).

The total number of possibilities is \( \sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2} \Rightarrow \)
\( d(n) = \frac{(n+1)(n+2)}{2} \).
Griffiths, Problem 4.57

(a) 
\[ [x, x] = [y, y] = 0; \quad [p_x, p_x] = [p_y, p_y] = 0; \]
\[ [x, p_y] = [y, p_x] = 0 \Rightarrow [q_1, q_2] = 0 \text{ and } [p_1, p_2] = 0 \]
\[ [q_1, p_1] = \frac{\hbar}{2i} \left[ x + \frac{a^2}{\hbar} (p_y - \frac{\hbar x}{a^2}) \right] \]
\[ = \frac{1}{2} \sum \left[ [x, p_x] - \frac{\hbar}{a^2} [x, y] + \frac{a^2}{\hbar} [p_y, p_x] - \frac{\hbar}{a^2} [p_y, y] \right] \]
\[ = \frac{1}{2} \sum \{ it_0 - (-it_0)^2 \} = it_0. \]
Similarly, \[ [q_2, p_2] = it_0. \]

(b) 
\[ q_1^2 = \frac{1}{2} \sum \{ x^2 + \frac{a^2}{\hbar} (p_y + p_y) \} \]
\[ q_2^2 = \frac{1}{2} \sum \{ x^2 - \frac{a^2}{\hbar} (p_y + p_y) \} \]
\[ \Rightarrow q_1^2 - q_2^2 = \frac{a^2}{\hbar} (p_y + p_y) \]
\[ p_1^2 = \frac{1}{2} \sum \{ p_x^2 - \frac{\hbar}{a^2} (p_y + p_y) + \frac{\hbar^2}{a^4} y^2 \} \]
\[ p_2^2 = \frac{1}{2} \sum \{ p_x^2 + \frac{\hbar}{a^2} (p_y + p_y) + \frac{\hbar^2}{a^4} y^2 \} \]
\[ \Rightarrow p_1^2 - p_2^2 = -\frac{\hbar}{a^2} (p_y + p_y) \]
\[ \Rightarrow \frac{\hbar}{2a^2} (q_1^2 - q_2^2) + \frac{a^2}{2\hbar} (p_1^2 - p_2^2) = \frac{1}{2} (x p_y + p_y x - y p_x - p_x y) \]

But \( p_y x = x p_y \) and \( p_y y = y p_x \) \( \Rightarrow \frac{1}{2} (2x p_y - 2y p_x) = x p_y - y p_x \).
(c) Part b  \[ L_2 = \left( \frac{\hbar^2}{2a^2} q_1^2 + \frac{a^2}{2\hbar} p_1^2 \right) - \left( \frac{\hbar^2}{2a^2} q_2^2 + \frac{a^2}{2\hbar} p_2^2 \right) \]

Each is of the form \( \frac{1}{2} m \omega^2 q^2 + \frac{1}{2} m \omega^2 p^2 \) with \( m = \frac{\hbar}{a^2}, \omega = 1 \)
as desired.

(d) Since \( \omega = 1 \) for both \( H_1 \) and \( H_2 \), their respective eigenvalues are \( E_{n_1} = \hbar (n_1 + \frac{1}{2}) \) and \( E_{n_2} = \hbar (n_2 + \frac{1}{2}) \).

\( L_2 \) has eigenvalues \( E_{n_1} - E_{n_2} = \hbar \left( n_1 + \frac{1}{2} - n_2 - \frac{1}{2} \right) = \hbar \left( n_1 - n_2 \right) \).

The eigenvalues of \( L_2 \) are \( n \hbar \), with \( n \) any integer.
Homework Set 4, Extra Problem 1

(a) For \( r \ll 1 \), \( n_e(x) \propto \frac{i}{r} \) is forbidden \( \Rightarrow \ \hat{R}_e(r) = \alpha_1 j_e(ikr) \).

(b) For \( r \to \infty \), \( j_e(ikr) \to \frac{\sin(ikr - \frac{\pi}{2})}{ikr} \)

\[ n_e(ikr) \to -\frac{\cos(ikr - \frac{\pi}{2})}{ikr} \]

Thus, \( \hat{h}_e^{(1)} = j_e(ikr) + in_e(ikr) \to \frac{\sin(ikr - \frac{\pi}{2}) - i\cos(ikr - \frac{\pi}{2})}{ikr} \)

\[ = -\frac{\cos(ikr - \frac{\pi}{2}) + i\sin(ikr - \frac{\pi}{2})}{ikr} \]

\[ = -\frac{e^{i(ikr - \frac{\pi}{2})}}{ikr} \]

has the correct behavior as \( r \to \infty \). Thus,

\( \hat{R}_e(r) = \delta h_e^{(1)}(ikr) \) solves Eq. 4.35 for \( r > a \).

(c) We need \( R \) and \( R' \) continuous \( \Rightarrow \alpha j_e(ka) = \delta h_e^{(1)}(ika) \)

and \( ikj_e(ika) = iKS h_e^{(1)}(ika) \). Dividing gives:

\[ \frac{\delta j_e(ika)}{j_e(ika)} = iK \frac{h_e^{(1)}(ika)}{h_e(ika)} \]
Homework Set #7, Extra Problem #2

(a) When \( E = V = 0 \), Eq. 4.35 reduces to \( \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = k(l+1)R \).

If we substitute \( R = r^{-\alpha} \), we find the most general solution is \( C r^l + \frac{D}{r^{l+1}} \). We must have \( C = 0 \) to have a bounded solution as \( r \to \infty \). Thus,

\[
R_r(r) = \frac{D}{r^{l+1}}, \quad \text{while } R_\ell(r) = A j_l(kr), \quad \text{from before.}
\]

(b) Continuity at \( r = a \) \( \Rightarrow \) \( A j_l(ka) = \frac{D}{a^{l+1}} \).

\[
A k j'_l(ka) = -\frac{(l+1) D}{a^{l+2}}
\]

Taking the ratio, we get \( \frac{k j'_l(ka)}{j_l(ka)} = \frac{(l+1) D}{a^{l+2}} \)

\[
\Rightarrow k a j'_l(ka) = -(l+1) j_l(ka) \Rightarrow (l+1) j_l(ka) + k a j'_l(ka) = 0.
\]

But the spherical Bessel recursion relation gives:

\[
\frac{l+1}{k a} j_l(ka) + j'_l(ka) = j_{l-1}(ka) \Rightarrow j_{l-1}(ka) = 0 \quad \text{as desired.} \]
Homework Set # 7, Extra Problem # 3

From Problem 4.38, we know there are \( d(n) \) degenerate eigenfunctions with energy \( E_n = \hbar \omega (n + \frac{3}{2}) \), and they can all be written in the form

\[
y_{ijk} (x,y,z) = N_{ijk} e^{-\frac{x^2 + y^2 + z^2}{2\hbar \omega}} H_i (ax) H_j (by) H_k (cz) \quad \text{with} \quad a = \sqrt{\frac{m_e}{\hbar}}
\]

where \( i + j + k = n \). Note that all such functions have parity \( (-1)^i (-1)^j (-1)^k \).

Let's focus on the Hermite polynomial terms, \( H_i (ax) H_j (by) H_k (cz) \), which is a sum of terms \( x^a y^b z^c \), where \( a \leq i, b \leq j, c \leq k \), and \( a, b, c \) are even/odd as \( i, j, k \) are even/odd. Note that all such terms obey \( a + b + c = n, n-2, n-4, \ldots, (0 \text{ or } 1) \).

\[\text{HOWEVER,}\]

We can also separate the 3-d harmonic oscillator in spherical polar coordinates. In this case, eigenfunctions with energy \( E_n = \hbar \omega (n + \frac{3}{2}) \) can be written in the form:

\[
y_{nlm} (r, \theta, \phi) = Y_{nlm} (\theta, \phi) e^{-\frac{r^2}{2\hbar \omega}} \text{ with } l = \sqrt{n + \frac{3}{2}}, \quad Y_{nlm} (\theta, \phi) = \frac{1}{r^{l+\frac{1}{2}}} \sqrt{\frac{2l+1}{4\pi}} H_l (\sqrt{2\hbar \omega} r) e^{-\frac{r^2}{2\hbar \omega}}
\]

We must be able to write \( y_{ijk} \) as linear combinations of \( y_{nlm} \) and \( y_{nlm} \) as linear combinations of \( y_{ijk} \). Furthermore, the degeneracy \( d(n) \) is a function only of \( E_n \). Thus, we must have the same number of linearly independent eigenfunctions in the two coordinate systems.
Relating the two sets of eigenfunctions:

The $\psi_{lm}$ have parity $(-1)^l$. But from above, they also have parity $(-1)^m$. Thus, $l$ must be even/odd as $m$ is even/odd.

Meanwhile, $\psi_{lm}^0 = r^l \psi_{l+1m}\,,\quad P(z,r)$, where $P(z,r)$ is a polynomial in $z$ of order $l-1m$ that is even/odd in $z$ as $l-1m$ is even/odd.

For example, from Table 4.3, if we ignore normalization constants,

then for $\psi_3^0$, $P(z,r) = 5z^3 - 3z^2$
and for $\psi_3^1$, $P(z,r) = 5z^2 - z^2$

Note that the $r^2$ terms multiply different factors in $\psi_3^0$ and $\psi_3^1$. Thus, they can't be moved into $R_{lm}$, which must be independent of $m$. Thus, we must follow the alternative path of replacing all of the $r^2$ factors in the $P(z,r)$ terms by $x^2 + y^2 z^2$. Once we make that substitution, we find that $\psi_{lm}$ can always be written as a sum of terms $x^a y^b z^c$ with $a + b + c = l$.

Equating the powers of $x, y, z$ that appear in $\psi_{ijk}$ and $\psi_{lm}$, we conclude that

$$l = n, n-2, n-4, \ldots, (0 or 1) \text{ as } n \text{ is even/odd}.$$
Did we get them all?

We know \( d(n) = \frac{(n+1)(n+2)}{2} \) in the Cartesian basis. In the spherical basis, we have \( d(n) = \sum_{\text{allowed \ values}} (2l+1) \). We can verify that the latter equals the former by induction:

\( n=0 \Rightarrow l=0, m=0 \), consistent with \( d(0) = \frac{1 \cdot 2}{2} = 1 \).

\( n=1 \Rightarrow l=1, m=-1,0,1 \), consistent with \( d(1) = \frac{2 \cdot 3}{2} = 3 \).

Unnecessary, but informative:

\( n=2 \Rightarrow l=0, m=0 \) or \( l=2, m=-2,-1,0,1,2 \), for a total degeneracy of \( 1+5 = 6 \). This is consistent with \( d(2) = \frac{3 \cdot 4}{2} = 6 \).

Induction from here:

The degeneracy of \( E_{n+2} = \frac{(n+3)(n+4)}{2} \). This is larger than \( d(n) \) by

\[ \Delta = \frac{(n+3)(n+4)}{2} - \frac{(n+1)(n+2)}{2} = \frac{4n+10}{2} = 2n+5 \overset{?}{=} 2(n+2)+1 \].

This is just the added degeneracy we get by adding \( l=n+2 \) to the previous set. It works!
Homework Set # 7, Extra Problem # 4

(a) \[ \frac{2}{3} c, a^2 \bar{c} = a^* a a + a a^* a = (a^* a + a^* a) a = \frac{2}{3} a, a a^* \bar{a} = a. \]
\[ \frac{2}{3} c, a^2 \bar{c} = a^* a a + a^* a a = a^* (a^* a + a a^*) = a^* \frac{2}{3} a, a a^* \bar{a} = a^*. \]

(b) \[ a | c \rangle = (C + a C) | c \rangle = C [a | c \rangle + a (c | c \rangle) = (C + c) [a | c \rangle \Rightarrow C [a | c \rangle] = (1 - c) [a | c \rangle \Rightarrow a | c \rangle \text{ is an eigenvector of } C \text{ with eigenvalue } 1 - c. \]
But \[ \langle c | a^* a | c \rangle = \langle c | C | c \rangle = c = | a | c \rangle |^2 \Rightarrow a | c \rangle = \sqrt{c} | 1 - c \rangle. \]
\[ a^+ | c \rangle = (C a^* + a^* c) | c \rangle = C [a^+ | c \rangle + a^* (c | c \rangle) \Rightarrow a^+ | c \rangle \text{ is also an eigenvector of } C \text{ with eigenvalue } 1 - c. \]
Meanwhile, \[ \langle c | a a^+ | c \rangle = \langle c | 1 - a^* a | c \rangle = 1 - c \Rightarrow a^+ | c \rangle = \sqrt{1 - c} | 1 - c \rangle. \]
(possibly \( e^{i \theta} \))

(c) \[ C^* = (a^* a)^* = a^* (a^*)^* = a^* a = C \Rightarrow C \text{ is hermitian.} \]
\[ C^2 = a^* a a^* a = a^* (1 - a^* a) a = a^* a - (a^* a)^2 = a^* a = C \Rightarrow C^2 = C. \]
\[ \Rightarrow C \text{ is a projection operator} \]

(d) \[ a^2 | c \rangle = 0 = a (a | c \rangle) = a (\sqrt{c} | 1 - c \rangle) = \sqrt{c} a (1 - c) | 1 - (1 - c) \rangle = \sqrt{c} (1 - c) | c \rangle \Rightarrow c (1 - c) = 0 \Rightarrow c = 0 \text{ or } c = 1. \]
Both must be present.