PHYSICS 412 – Homework Set #5
Due in class, Friday, February 23

Note: This problem set is unlikely to be graded and returned before Exam 1. So if you want access to your solutions while studying, make yourself copies before you turn them in.

Do the following seven problems from Griffiths:

2.31, pg 83
2.34, pg 83-84
   Skip part (c). We found an equivalent – but more general – formula in class.
2.51, pg 89-90
   Don’t bother to normalize $\psi_0$ in part (b). Also don’t bother to graph it.
   Hint: You don’t need to derive the answers given. Just show that they work.
2.53, pg 91-92
3.21, pg 123
3.23, pg 124
3.34, pg 127
   To minimize the degrees of freedom that you need to consider, assume the amplitude for $\psi_0$ is real and positive at $t = 0$.

Also do the following additional problem:

A particle of mass $m$ is moving in one dimension under the influence of a potential function given by:

$$ V(x) = \begin{cases} 
+\infty & \text{for } x < 0 \\
-V_0 & \text{for } 0 < x < a \\
0 & \text{for } a < x 
\end{cases} $$

(where $V_0$ is a positive constant)

(a) Give as many details as you possibly can about the energy eigenfunctions of this system that have $E_i < 0$. Things to consider include the mathematical forms of $\psi(x)$ in different regions, boundary conditions, eigenvalue conditions, number of bound states, etc.

(b) Now assume that $V_0$ is just barely large enough for there to be 3 bound states for this system. Draw qualitative figures on the attached sheet of graphs showing what the (real-valued) eigenfunctions associated with these 3 bound states must look like.
Ground state

First-excited state

Second-excited state
Griffiths, Problem 2.31

In class, we saw the finite square well has only one bound state when \( \frac{2mV_0a^2}{\hbar^2} \leq \left( \frac{\pi}{2} \right)^2 \). To go to the \( S \)-\( F \) limit, we need:

\[
V_0 \to \infty, \quad a \to 0, \quad 2V_0a = x = \text{constant} \implies \frac{2mV_0a^2}{\hbar^2} \to 0 < \left( \frac{\pi}{2} \right)^2, \quad \text{the } S \text{-} F \text{ potential is weak.}
\]

In class, we saw the even eigenfunction for the square well obeys:

\[
ka \tan ka = ka, \quad \text{with } ka = \sqrt{\frac{2m(V_0 - |E|)a^2}{\hbar^2}}.
\]

\[
ka = \sqrt{\frac{2m|E|a^2}{\hbar^2}}.
\]

\[
\implies \left( ka \right)^2 + \left( ka \right)^2 = \frac{2mV_0a^2}{\hbar^2} \to 0. \quad \text{Thus, we can use small-angle approximations:}
\]

\[
\left( ka \right)^2 = ka \implies \left( V_0 \gg |E| \right) \quad \frac{2mV_0a^2}{\hbar^2} = \frac{m a^2}{\hbar^2} = \sqrt{\frac{2m|E|a^2}{\hbar^2}}.
\]

\[
\implies E = -\frac{m a^2}{2\hbar^2} \left( = - \frac{m a^2}{2\hbar^2} \right), \quad \text{consistent with Eq. 2.129.}
\]

Again, \( V_0a^2 \to 0 \), so we can use small-angle approximations in Eq. 2.169 \( \Rightarrow \)

\[
\frac{1}{T} = 1 + \frac{V_0}{4E(E + V_0)} \quad \frac{4a^2}{\hbar^2} \left( 2m(E + V_0) \right) = 1 + \frac{m(4V_0a^2)}{2\hbar^2 E}
\]

\[
= 1 + \frac{m a^2}{2\hbar E} \implies T = \frac{1}{1 + \frac{m a^2}{2\hbar E}}, \quad \text{as given in Eq. 2.141.}
\]
Griffiths, Problem 2.34

(a) \( V = V_0 \)

\[ y = \frac{1}{E} \begin{cases} 
D \exp(-kx) & \text{if } k^2 > \frac{mE}{2} \\
0 & \text{if } k^2 \leq \frac{mE}{2} 
\end{cases} \]

WLOG, since it's just a scale factor

\[ y_c = e^{-ikx} + Be^{-ikx} \text{ with } E = \frac{\hbar^2 k^2}{2m} \]

(we set A = 1 for convenience)

Continuity of \( y \) at \( x = 0 \) \( \Rightarrow \) (1) \( 1 + B = D \)

Continuity of \( y' \) at \( x = 0 \) \( \Rightarrow \) (2) \( ik(1-B) = -kD \)

\[ \frac{ik(1-B)}{1+B} = -k \Rightarrow \frac{ik(1-B)}{ik-k} = -k \]

\[ \Rightarrow B = \frac{ik+k}{ik-k} \]

Thus, \( |B| = 1 \) \( \Rightarrow \) \( R = |B|^2 = 1 \) as expected since \( x > 0 \) is classically forbidden.

(b) \( y_c \) is unchanged, but now \( y_c = Ce^{-ikx} \) with \( \frac{\hbar^2 k^2}{2m} = E - V_0 \)

Continuity of \( y \) at \( x = 0 \) \( \Rightarrow \) (1) \( 1 + B = C \)

Continuity of \( y' \) at \( x = 0 \) \( \Rightarrow \) (2) \( ik(1-B) = ik'C \)

\[ \frac{ik(1-B)}{1+B} = k' \Rightarrow k - kB = k' + k'B \Rightarrow \]

\[ B = \frac{k-k'}{k+k'} \]

\[ C = \frac{k+k' + k-k'}{k+k'} = \frac{2k}{k+k'} \]

\[ R = \left| \frac{k-k'}{k+k'} \right|^2 = \left( 1 - \frac{E-V_0}{E} \right)^2 \]

\[ R = \left( \frac{1 - \sqrt{E-V_0}}{1 + \sqrt{E-V_0}} \right)^2 \]
(d) \[ T = \frac{v^{2}}{u} \left| C \right|^{2} = \frac{k'}{k} \frac{4k'^{2}}{(k+k')^{2}} = \frac{4kk'}{(k+k')^{2}} \]

\[ R + T = \frac{(k-k')^{2}}{(k+k')^{2}} + \frac{4kk'}{(k+k')^{2}} = \frac{k^{2} - 2kk' + k'^{2} + 4kk'}{(k+k')^{2}} \]

\[ = \frac{(k+k')^{2}}{(k+k')^{2}} \left[ 1 \right] \text{ as expected.} \]
Griffiths, Problem 2.51

(b) \( \psi_0 = \frac{A}{\cosh(ax)} \) \( \Rightarrow \) \( \psi_0' = -\frac{Aa \sinh(ax)}{\cosh^2(ax)} \) \( \Rightarrow \) \( \psi_0'' = -Aa^2 \frac{1 - \sinh^2(ax)}{\cosh^3(ax)} \)

Thus, \( H \psi_0 = -\frac{\hbar^2}{2m} \frac{d^2 \psi_0}{dx^2} + V(x) \psi_0 \)

\[ = A \left[ + \frac{\hbar^2 a^2}{2m} \frac{1 - \sinh^2(ax)}{\cosh^3(ax)} - \frac{\hbar^2 a^2}{m} \frac{1}{\cosh^3(ax)} \left( \frac{2}{2} \right) \right] \]

\[ = -A \frac{\hbar^2 a^2}{2m} \frac{\sinh(ax) + 1}{\cosh^3(ax)} \]

\[ = -\left( \frac{\hbar^2 a^2}{2m} \right) \frac{A}{\cosh(ax)} = -\left( \frac{\hbar^2 a^2}{2m} \right) \psi_0 \]

\[ \Rightarrow \psi_0 \text{ is an eigenfn with } E = -\frac{\hbar^2 a^2}{2m}. \psi_0 \text{ has no modes, so it must be the ground state.} \]

(c) Write Schrodinger's equation as:

\[ \frac{d^2 \psi}{dx^2} + \frac{2m}{\hbar^2} (E - V(x)) \psi = 0 \]

with \( E = \frac{\hbar^2 k^2}{2m} \) \( \Rightarrow \frac{d^2 \psi}{dx^2} + \left( \frac{k^2}{\cosh^2(ax)} \right) \psi = 0 \)

To show that \( \psi_k \) solves this, first factor off the \( \frac{A}{\cosh(ax)} \), which is just a normalization factor. Write the rest of \( \psi_k \) as:

\[ \psi_k = \left[ e^{ik - a \tanh(ax)} \right] e^{ikx} = f(x) e^{ikx}. \]

Then \( \frac{d^2 \psi_k}{dx^2} = f''(x) e^{ikx} + 2ikf'(x) e^{ikx} - k^2 f(x) e^{ikx}. \)

If we substitute this into \( \odot \) above, then cancel the \( e^{ikx} \) factors, we're left with:

\[ f''(x) + 2ikf'(x) + 2a^2 \sech^2(ax) f = 0 \]

Does this work? \( \frac{df}{dx} = \frac{a^2}{\cosh^2(ax)} \Rightarrow \frac{d^2 f}{dx^2} = \frac{2a^2 \sinh(ax)}{\cosh^3(ax)} \)
Substituting gives:

\[
2a^3 \frac{\sinh(\alpha x)}{\cosh^2(\alpha x)} - 2i\alpha^2 \frac{\cosh^2(\alpha x)}{\cosh^2(\alpha x)} + 2a^2 \frac{(ik - a) \frac{\sinh(\alpha x)}{\cosh(\alpha x)}}{\cosh(\alpha x)} = 0.
\]

It works! \( \phi_k \) is an eigenfunction with eigenvalue \( E = \frac{k^2 \hbar^2}{2m} \).

For large positive \( k \), \( \tanh(\alpha x) \to +1 \to \)

\[
\phi_k(x) \to A \left( \frac{ik - a}{ik + a} \right) e^{ikx}.
\]

\[
\left| \frac{ik - a}{ik + a} \right| = 1 \Rightarrow |F| = |A| \Rightarrow T = \frac{|F|^2}{|A|^2} = 1, \text{ as expected.}
\]

Since \( B = 0 \) \( \Rightarrow R = 0 \).
Griffiths, Problem 2.53

We saw in class that $S$ is both unitary and symmetric. If $S = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, then $S^T = S^{-1} = \begin{pmatrix} a^* & b^* \\ b^* & c^* \end{pmatrix}$ with $|a| = |c|$

(a) We have $B = aA + bG \Rightarrow G = \frac{1}{b}B - \frac{a}{b}A$
$F = bA + cG \Rightarrow F = bA + \frac{c}{b}B - \frac{ac}{b}A$

$M = \begin{pmatrix} b^2-ac & c/b \\ -a/b & c/b \\ -a/b & 1/b \end{pmatrix}$

Note (not required): $\text{det}(M) = 1$ and $\frac{b^2-ac}{b} = \frac{1}{b}$ (from $S^{-1} = S^*$)

To go back, write $M_2 = \frac{c}{b}$, $M_2 = -\frac{a}{b}$, $M_2 = \frac{1}{b}$

$S_{11} = a = -\frac{M_{21}}{M_{22}} \quad S_{12} = S_{21} = b = \frac{1}{M_{22}} \quad S_{22} = c = \frac{M_{12}}{M_{22}}$

Then $R_e = |S_{11}|^2 = \frac{|M_{21}|^2}{M_{22}} \quad T_e = |S_{21}|^2 = \frac{1}{|M_{22}|^2}$
$R_f = |S_{22}|^2 = \frac{|M_{12}|^2}{M_{22}} \quad T_f = |S_{12}|^2 = \frac{1}{|M_{22}|^2}$

(b) $Ae^{i\epsilon} + Be^{-i\epsilon} \quad M_1 \quad Fe^{i\epsilon} + Ge^{-i\epsilon} \quad ML \quad Le^{i\epsilon} + Me^{-i\epsilon}$

\[ V_1 \]

and find $G$ would go to $V_2$.

Without $V_2$, we would have $M = M_1^4$. Without $V_1$, we would have $M_2$, and $F$ and $G$ would go to $-\infty$.

Thus, $M = M_2 M_1 \begin{pmatrix} A \\ B \end{pmatrix} \Rightarrow \boxed{M = M_2 M_1}$ as desired.
(c) If we maintain Eq. 2.131 + 2.132, but move the $S$-field from $x=0$ to $x'=+a$, then:
Eq. 2.133 becomes

\[ Fe^{ika} + Ge^{-ika} = Ae^{ika} + Be^{-ika} \]

Similarly, Eq. 2.135 becomes:

\[ Fe^{-ika} - Ge^{ika} = (1 + 2i\beta)e^{ika} A - (1 - 2i\beta)e^{-ika} B \]

\[ \frac{e^{-ika}}{2} \cdot (1 + 2) \Rightarrow F = (1 + i\beta)A + i\beta e^{+2ika} B \]

\[ \Rightarrow M_{11} = 1 + i\beta \quad M_{12} = i\beta e^{-2ika} \]

\[ \frac{e^{ika}}{2} \cdot (1 - 2) \Rightarrow G = -i\beta e^{-2ika} A + (1 - i\beta) B \]

\[ \Rightarrow M_{21} = -i\beta e^{2ika} \quad M_{22} = 1 - i\beta \]

(d) The $M$ in part c is $M_2$. We obtain $M_1$ by replacing $a \rightarrow -a$.

\[ M = \begin{pmatrix} 1 + i\beta & i\beta e^{-2ika} \\ -i\beta e^{2ika} & 1 - i\beta \end{pmatrix} \begin{pmatrix} 1 + i\beta & i\beta e^{2ika} \\ -i\beta e^{-2ika} & 1 - i\beta \end{pmatrix} \]

\[ = \begin{pmatrix} 1 - \beta^2 + 2i\beta + \beta^2 e^{4ika} & (i\beta - \beta^2) e^{2ika} + (i\beta + \beta^2) e^{-2ika} \\ (-i\beta + \beta^2) e^{2ika} + (-i\beta - \beta^2) e^{-2ika} & \beta e^{2ika} + 1 - \beta^2 - 2i\beta \end{pmatrix} \]

The transmission coef. $T = \frac{1}{1 - 2i\beta + \beta^2 (e^{ika} - 1)^2}$
Griffiths, Problem 3.21

\( \hat{P}^2 = |\alpha \rangle \langle \alpha| = |\alpha \rangle \langle \alpha| |\alpha \rangle \langle \alpha| = |\alpha \rangle \langle \alpha| \), hence \( \langle \alpha|\alpha \rangle = 1 \Rightarrow \)

\( \hat{P}^2 = \hat{P} \).

Let \( |\nu \rangle \) be an eigenvector, with eigenvalue \( \nu \).

Then \( \hat{P}^2 |\nu \rangle = \hat{P}(\nu |\nu \rangle) = \nu^2 |\nu \rangle = \hat{P} |\nu \rangle = \nu |\nu \rangle \)

\( \Rightarrow \nu^2 = \nu \Rightarrow \boxed{\nu = 0, 1} \) are the allowed eigenvalues.

Clearly, \( \hat{P} |\alpha \rangle = |\alpha \rangle \langle \alpha| |\alpha \rangle = |\alpha \rangle \Rightarrow \)

|\alpha \rangle \) is an eigenvector with eigenvalue \( 1 \).

Meanwhile, any \( |\beta \rangle \) orthogonal to \( |\alpha \rangle \) will obey \( \hat{P} |\beta \rangle = |\alpha \rangle \langle \alpha| |\beta \rangle = 0 = 0 |\beta \rangle \Rightarrow \)

All vectors orthogonal to \( |\alpha \rangle \) are eigenvectors with eigenvalue \( 0 \).
Griffiths, Problem 3.23

From class, \( \hat{H} = \sum_{i,j} H_{ij} \hat{c}_i \hat{c}_j \). Reading them off, this means the matrix \( H = \begin{pmatrix} E & E \\ E & -E \end{pmatrix} \). To find the eigenvalues, we need

\[
\det(H - \lambda I) = (E - \lambda)(-E - \lambda) - E^2 = 0 \Rightarrow \lambda^2 - 2E^2 = 0
\]

\[\Rightarrow \lambda = \pm \sqrt{2} E \]

Eigenvector for \( \lambda = +\sqrt{2} E \):

\[
H \begin{pmatrix} a \\ b \end{pmatrix} = \sqrt{2} E \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow Ea + Eb = \sqrt{2} E a \Rightarrow b = (\sqrt{2} - 1)a
\]

Check: \( Ea - Eb = \sqrt{2} E b \)

\[\Rightarrow a - \sqrt{2} a + a = (2 - \sqrt{2})a = \sqrt{2} (\sqrt{2} - 1)a = \sqrt{2} b \]

Normalization \( 1 = a^2 + b^2 = a^2 + (\sqrt{2} - 1)^2 a^2 = a^2 \left[ 1 + 2 - 2\sqrt{2} + 1 \right] \)

\[= (4 - 2\sqrt{2}) a^2 \Rightarrow a = \frac{1}{\sqrt{4 - 2\sqrt{2}}} \text{, } b = \frac{\sqrt{2} - 1}{\sqrt{4 - 2\sqrt{2}}} \]

\[\Rightarrow \text{The eigenvector is } \frac{1}{\sqrt{4 - 2\sqrt{2}}} \begin{pmatrix} 1 \\ \sqrt{2} - 1 \end{pmatrix}, \text{ but that's okay.} \]

Eigenvector for \( \lambda = -\sqrt{2} E \):

\[
H \begin{pmatrix} a \\ b \end{pmatrix} = -\sqrt{2} E \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow Ea + Eb = -\sqrt{2} E a \Rightarrow b = (\sqrt{2} + 1)a
\]

Check: \( Ea - Eb = -\sqrt{2} E b \Rightarrow a + (\sqrt{2} + 1)a = (\sqrt{2} + 2)a = (-\sqrt{2})(\sqrt{2} + 1)a \)
Normalization: \[ 1 = a^2 + b^2 = a^2 + (\sqrt{2}+1)^2 \]

\[ = a^2 + (2 + 2\sqrt{2} + 1) a^2 = (4 + 2\sqrt{2}) a^2 \Rightarrow \]

\[ a = \frac{1}{\sqrt{4 + 2\sqrt{2}}} , \quad b = -(\sqrt{2}+1)b = \frac{\sqrt{2}+1}{\sqrt{4 + 2\sqrt{2}}} \]

\[ \Rightarrow \text{The eigenvector is} \quad \begin{pmatrix} 1 \\ -\frac{\sqrt{2}+1}{\sqrt{4 + 2\sqrt{2}}} \end{pmatrix} \]

Cross-check (not required):

The two eigenvectors should be orthogonal. The radicals are messy. Rather than take time to simplify them algebraically,

I plugged: \[ \frac{1}{\sqrt{4 - 2\sqrt{2}}} , \quad \frac{1}{\sqrt{4 + 2\sqrt{2}}} + \frac{\sqrt{2} - 1}{\sqrt{4 - 2\sqrt{2}}} - \frac{(\sqrt{2}+1)}{\sqrt{4 + 2\sqrt{2}}} \]

into MathCad. I got back zero, as desired.
Griffiths, Problem 3.34

To measure $\frac{1}{2}$ $\psi_0$ or $\frac{3}{2}$ $\psi_0$ with equal probability, we must have:

$$\Psi(x, t) = c_0 \psi_0(x) e^{-i(\frac{1}{2} \omega t)} + c_1 \psi_1(x) e^{-i(\frac{3}{2} \omega t)}$$

with

$$|c_0|^2 = |c_1|^2 = \frac{1}{2} \Rightarrow c_0 = \frac{1}{\sqrt{2}} ; c_1 = \frac{e^{\frac{i\phi}{2}}}{\sqrt{2}} , \text{ with } \phi \text{ a real constant.}$$

Then $<p(t)> = \frac{1}{2} <\psi_0 | p | \psi_1> e^{-i(\frac{1}{2} \omega + \frac{3}{2} \omega) t}$

$$+ \frac{1}{2} <\psi_1 | p | \psi_0> e^{-i(\frac{1}{2} \omega - \frac{3}{2} \omega) t} , \text{ where I used }<\psi_0 | p | \psi_0> =<\psi_1 | p | \psi_1> = 0.$$

$$<\psi_m | p | \psi_n> = i \sqrt{\frac{\omega t}{2}} \left( \delta_{mn+1} - \delta_{mn-1} \right)$$

$$\Rightarrow <p(t)> = \sqrt{\frac{\omega t}{2}} \left[ -i e^{-i(\omega t - \phi)} + i e^{i(\omega t + \phi)} \right]$$

$$= -\sqrt{\frac{\omega t}{2}} \sin(\omega t - \phi) . \text{ So...}$$

The largest value of $<p>$ is $\sqrt{\frac{\omega t}{2}}$, and it occurs

at $t = 0$ when $\phi = \frac{\pi}{2} \Rightarrow$

$$\Psi(x, t) = \frac{1}{\sqrt{2}} \left[ \psi_0 e^{-\frac{i}{2} \omega t} + i \psi_1 e^{-\frac{3}{2} i \omega t} \right]$$.
Homework Set 5

Extra Problem

(a)

\[ V \rightarrow \infty \]

\[ V = 0 \]

\[ V = -V_0 \]

\[ x = 0 \quad x = a \]

\[ V = +\infty \text{ for } x < 0 \Rightarrow 4(x) = 0 \text{ for } x < 0 \]

For \( 0 < x < a \), Sch. eq. \[
\frac{d^2y}{dx^2} + \frac{2m}{\hbar^2}(E_i - (-V_0))y = 0
\]

\[ = \frac{d^2y}{dx^2} + \frac{2m}{\hbar^2}(V_0 - 1E_i)\]

\[ 4(x) = A \cos(kx) + B \sin(kx) \]

with \[ k = \sqrt{\frac{2m}{\hbar^2}(V_0 - 1E_i)} \]. But \[ 4(0) = 0 \Rightarrow 4 = B \sin(kx) \text{ for } 0 < x < a \]

For \( a < x \), Sch. eq. \[
\frac{d^2y}{dx^2} - \frac{2mE_i}{\hbar^2}y = 0
\]

\[ \Rightarrow 4 = De^{-kx} \text{ with } k = \sqrt{\frac{2mE_i}{\hbar^2}} \]

At \( x = a \), we need \( 4 \) and \( 4' \) continuous \[
B \sin(ka) = De^{-ka}
\]

\[ B \cos(ka) = -kDe^{-ka} \]

\[ \Rightarrow D = Be^{ka} \sin(ka) \text{ and } k \cos(ka) = -k \]

or \[ k \cos(ka) = -ka \text{ (dimensionless)} \]

This is exactly the same eigenvalue condition that we had in class for the odd eigenfunctions of the finite square well.
$ka$ must be $\geq 0$, so $\cot(ka) \leq 0$ for there to exist a solution. This means $\frac{\pi}{2} < ka < \pi$, $\frac{3\pi}{2} < ka < 2\pi$, $\frac{5\pi}{2} < ka < 3\pi$, ...

$\Rightarrow$ No solution for $k_{max}a < \frac{\pi}{2}$

One solution for $\frac{\pi}{2} < k_{max}a < \frac{3\pi}{2}$

Two solutions for $\frac{3\pi}{2} < k_{max}a < \frac{5\pi}{2}$, ...

with $k_{max}a = \sqrt{\frac{2mV_0}{k_1^2a}}$

Finally, B is determined (up to an overall complex phase factor) by the normalization condition:

$$\int_{-\infty}^{\infty} 141^2 dx = \int_{0}^{\infty} 141^2 dx = 1$$