Physics 606 – Mid-Term Exam

Wednesday, March 11, 2009

There are four questions on this exam. Each question is worth a total of 25 points. Please use only one side of each sheet of paper, start each problem on a new page, and encircle your final answer(s). Note: You may assume that all explicit wavefunctions given on this exam are normalized.

1. A particle with mass m and total energy $E = 0$ is moving in one dimension under the influence of a potential given by:

$$V(x) = V_0(1 - \frac{x^2}{a^2})$$

where $V_0 > 0$. Assume the particle is incident from the region $x \rightarrow -\infty$. Find the approximate transmission coefficient for the particle to end up in the region $x > a$. What condition(s) need to hold for your approximation to be valid?

2. A particle of mass m is moving in free space in one dimension. At $t = 0$, the particle's wavefunction is given by:

$$\psi(x,t=0) = \begin{cases} \sqrt{\dfrac{15}{16a^5}}(a^2 - x^2) & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases}$$

(a) Find the expectation values $\langle x \rangle$, $\langle x^2 \rangle$, $\langle p_x \rangle$, and $\langle p_x^2 \rangle$, all at time $t = 0$. Indicate which of these, if any, you would expect to change as functions of time, and why.
(b) What is the momentum wavefunction $\varphi(p_x, t)$ of the particle?

3. In the following two cases, the wavefunction of a particle of mass m is given at $t = 0$. For each case, determine the expectation values of $\langle x \rangle$ and $\langle E \rangle$, both as functions of time.
(a) A particle moving in an infinite square well, given by $V(x) = 0$ for $|x| < a$, $V(x) = +\infty$ for $|x| > a$, has wavefunction:

$$\psi(x,t=0) = \sqrt{\frac{1}{4a}}\cos(\frac{\pi x}{2a}) + \sqrt{\frac{3}{4a}}\sin(\frac{\pi x}{a}) \qquad \text{for } |x| < a$$

(b) A particle is moving in a one-dimensional harmonic oscillator potential, $V(x) = ½\, m\omega^2 x^2$, with wavefunction:

$$\psi(x,t=0) = \sqrt{\frac{4\alpha}{3\sqrt{\pi}}}(\alpha x)^2 \exp(-\frac{1}{2}(\alpha x)^2), \qquad \text{where } \alpha = \sqrt{\frac{m\omega}{\hbar}}$$

4. A particle of mass m is moving in a one-dimensional infinite square well, given by $V(x) = 0$ for $0 < x < 2a$, $V(x) = +\infty$ otherwise. The particle is acted on by an additional delta-function potential $V(x) = g\,\delta(x-a)$, where $g > 0$. (**NOTE**: As given, this potential is NOT centered on $x = 0$.)
(a) Find the equation(s) that determine the energy eigenvalues.
(b) Draw qualitative figures that indicate the shapes of the wavefunctions for the ground and first three excited states. Assume the dimensionless parameter $mga/\hbar^2$ is neither large, nor small.
(c) For the case of very small g, find the ground state energy, correct to leading non-vanishing order in the small dimensionless parameter $mga/\hbar^2$.
(d) For the case of very large g, find the ground state energy, correct to leading non-vanishing order in the small dimensionless parameter $\hbar^2/mga$.

Mid-Term Exam, Problem 1

The turning points are at $x = \pm a$, so the WKB approx gives:

$$T \sim \exp\left\{-2\int_{-a}^{a}\sqrt{\frac{2mV_0}{\hbar^2}\left(1-\frac{x^2}{a^2}\right)}\,dx\right\}$$

$$= \exp\left\{-4\int_{0}^{a}\sqrt{\frac{2mV_0}{\hbar^2}\left(1-\frac{x^2}{a^2}\right)}\,dx\right\}$$

$$\int_0^a \sqrt{1-\frac{x^2}{a^2}}\,dx \overset{x=a\sin\theta}{=} \int_0^{\frac{\pi}{2}}\sqrt{1-\frac{\cancel{a^2}\sin^2\theta}{\cancel{a^2}}}\;a\cos\theta\,d\theta = a\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta = \frac{\pi a}{4}$$

$$\Rightarrow T \sim \exp\left\{-\cancel{4}\,\frac{\pi a}{\cancel{4}}\sqrt{\frac{2mV_0}{\hbar^2}}\right\} = \exp\left\{-\pi\sqrt{\frac{2mV_0a^2}{\hbar^2}}\right\}$$

For validity, we need $\left|\frac{K'}{K^2}\right| \ll 1$ over most of $-a<x<a$

$$K = \sqrt{\frac{2mV_0}{\hbar^2}\left(1-\frac{x^2}{a^2}\right)} \Rightarrow \frac{K'}{K^2} = \frac{\left(\frac{1}{\cancel{2}}\right)\frac{1}{K}\,\frac{2mV_0}{\hbar^2}\left(-\frac{\cancel{2}x}{a^2}\right)}{K^2}$$

$$= \frac{\cancel{\frac{2mV_0}{\hbar^2}}\left(\frac{-x}{a^2}\right)}{K\left(\cancel{\frac{2mV_0}{\hbar^2}}\right)\left(1-\frac{x^2}{a^2}\right)} \Rightarrow \left|\frac{K'}{K^2}\right| = \frac{\frac{|x|}{a}}{(Ka)\left(1-\frac{x^2}{a^2}\right)}$$

(all three terms are dimensionless). $\frac{\frac{|x|}{a}}{\left(1-\frac{x^2}{a^2}\right)} = \frac{2}{3}$ for $\frac{|x|}{a}=\frac{1}{2}$,

1 for $\frac{|x|}{a}=\frac{1}{\sqrt{2}}$

$\to\infty$ for $|x|\to a$

$\Rightarrow$ Want $Ka \gg 1 \Rightarrow$ Need $\frac{mV_0a^2}{\hbar^2}\gg 1$

($\Leftrightarrow$ a wide, ~~deep~~ tall barrier)

Mid-Term Exam, Problem 2

(a) $\psi(x)$ is an even fn of x. Thus, $\boxed{\langle x\rangle = 0}$ and $\boxed{\langle p_x\rangle = 0}$, $\boxed{\text{ind of time}}$

$$\langle x^2\rangle = \int_{-a}^{a}\frac{15}{16a^5}(a^2-x^2)x^2(a^2-x^2)\,dx = \frac{15}{8a^5}\int_0^a(a^4-2a^2x^2+x^4)x^2\,dx$$

$$= \frac{15}{8a^5}\left[\frac{a^4x^3}{3}-\frac{2a^2x^5}{5}+\frac{x^7}{7}\right]_{x=0}^{a} = \frac{15}{8a^5}a^7\left(\frac{1}{3}-\frac{2}{5}+\frac{1}{7}\right)$$

$$= \frac{\cancel{15}}{8}a^2\left(\frac{35-42+15}{\cancel{105}}\right) = \frac{a^2}{8}\,\frac{8}{7} = \boxed{\frac{a^2}{7}}$$

The wave packet will spread, so this will $\boxed{\text{increase with } t}$

$$\langle p_x^2\rangle = \int_{-a}^{a}\frac{15}{16a^5}(a^2-x^2)\left(-\hbar^2\frac{d^2}{dx^2}\right)(a^2-x^2)\,dx$$

$$= \frac{15}{\cancel{16}a^5}(\cancel{2}\hbar^2)\cancel{2}\int_0^a(a^2-x^2)\,dx = \frac{15\hbar^2}{4a^5}\left[a^2x-\frac{x^3}{3}\right]_{x=0}^{a}$$

$$= \frac{\cancel{15}}{\cancel{4}}\frac{\hbar^2}{a^5}\frac{\cancel{2}}{\cancel{3}}a^3 = \boxed{\frac{5}{2}\frac{\hbar^2}{a^2}}$$

. Particle is free $\Rightarrow$ $\boxed{\text{ind of } t}$

(b) $$\phi(p_x,t=0) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty}e^{-i\frac{p_x x}{\hbar}}\psi(x,t=0)\,dx$$

ψ is even, $e^{-i\frac{p_x x}{\hbar}} = \cos\left(\frac{p_x x}{\hbar}\right) - i\sin\left(\frac{p_x x}{\hbar}\right) \Rightarrow$ imag part $=0$; real gives

$$\phi(p_x,t=0) = \frac{2}{\sqrt{2\pi\hbar}}\sqrt{\frac{15}{16a^5}}\int_0^a\cos\left(\frac{p_x x}{\hbar}\right)(a^2-x^2)\,dx$$

INT #329

$$= \sqrt{\frac{15}{8\pi\hbar a^5}}\left\{\left[\frac{a^2\sin\left(\frac{p_x x}{\hbar}\right)}{\frac{p_x}{\hbar}}\right]_{x=0}^{a} - \left[\frac{2x\cos\left(\frac{p_x x}{\hbar}\right)}{\left(\frac{p_x}{\hbar}\right)^2} + \frac{\left(\frac{p_x}{\hbar}\right)^2x^2-2}{\left(\frac{p_x}{\hbar}\right)^3}\sin\left(\frac{p_x x}{\hbar}\right)\right]_{x=0}^{a}\right\}$$

$$= \sqrt{\frac{15}{8\pi\hbar a^5}}\left\{\cancel{\frac{\hbar a^2}{p_x}\sin\left(\frac{p_x a}{\hbar}\right)} - \frac{2a\hbar^2}{p_x^2}\cos\left(\frac{p_x a}{\hbar}\right) - \cancel{\frac{\hbar a^2}{p_x}\sin\left(\frac{p_x a}{\hbar}\right)} + \frac{2\hbar^3}{p_x^3}\sin\left(\frac{p_x a}{\hbar}\right)\right\}$$

$$= \sqrt{\frac{15}{\cancel{8}\pi\hbar a^5}}\,\frac{\cancel{2}\hbar^2}{p_x^2}\left\{\frac{\hbar}{p_x}\sin\left(\frac{p_x a}{\hbar}\right) - a\cos\left(\frac{p_x a}{\hbar}\right)\right\}$$

$$\boxed{= \sqrt{\frac{15a}{2\pi\hbar}}\left(\frac{\hbar^2}{p_x^2a^2}\right)\left\{\frac{\sin\left(\frac{p_x a}{\hbar}\right)}{\frac{p_x a}{\hbar}} - \cos\left(\frac{p_x a}{\hbar}\right)\right\}}$$

$$\phi(p_x,t) = \phi(p_x,t=0)\,e^{-i\frac{E(p_x)t}{\hbar}} \quad\text{with}\quad E(p_x) = \frac{p_x^2}{2m}$$

$$\Rightarrow \boxed{\phi(p_x,t) = \sqrt{\frac{15a}{2\pi\hbar}}\left(\frac{\hbar^2}{p_x^2a^2}\right)\left\{\frac{\sin\left(\frac{p_x a}{\hbar}\right)}{\frac{p_x a}{\hbar}} - \cos\left(\frac{p_x a}{\hbar}\right)\right\}e^{-i\frac{p_x^2 t}{2m\hbar}}}$$

– or –

$$\boxed{\phi(p_x,t) = \sqrt{\frac{15a}{2\pi\hbar}}\left(\frac{\hbar}{p_x a}\right)^3\left\{\sin\left(\frac{p_x a}{\hbar}\right) - \left(\frac{p_x a}{\hbar}\right)\cos\left(\frac{p_x a}{\hbar}\right)\right\}e^{-i\frac{p_x^2 t}{2m\hbar}}}$$

Mid-Term Exam, Problem 3

(a) For the square well, $\psi_0(x) = \frac{1}{\sqrt{a}} \cos\left(\frac{\pi x}{2a}\right)$, with $E_0 = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 \pi^2}{8 m a^2}$

$$\psi_1(x) = \frac{1}{\sqrt{a}} \sin\left(\frac{\cancel{2}\pi x}{\cancel{2}a}\right) \text{ with } E_1 = \frac{\hbar^2\pi^2}{2ma^2}$$

Thus, $\psi(x,t=0) = \sqrt{\frac{1}{4}}\,\psi_0 + \sqrt{\frac{3}{4}}\,\psi_1$.

$$\langle E \rangle = \frac{1}{4}E_0 + \frac{3}{4}E_1 = \frac{1}{4}\,\frac{\hbar^2\pi^2}{8ma^2} + \frac{3}{4}\,\frac{\hbar^2\pi^2}{2ma^2} = \boxed{\frac{13}{32}\,\frac{\hbar^2\pi^2}{ma^2}\text{, ind of } t}$$

$$\langle x \rangle_t = \int_{-a}^{a} \left(\sqrt{\tfrac{1}{4}}\,\psi_0 e^{-i\frac{E_0 t}{\hbar}} + \sqrt{\tfrac{3}{4}}\,\psi_1 e^{-i\frac{E_1 t}{\hbar}}\right)^{*} x \left(\sqrt{\tfrac{1}{4}}\,\psi_0 e^{-i\frac{E_0 t}{\hbar}} + \sqrt{\tfrac{3}{4}}\,\psi_1 e^{-i\frac{E_1 t}{\hbar}}\right) dx$$

The diagonal terms are zero. The non-diagonal terms give:

$$\langle x \rangle_t = \frac{\sqrt{3}}{4}\left\{\int_{-a}^{a} \psi_0 x \psi_1\,dx\, e^{i\frac{(E_0-E_1)}{\hbar}t} + \int_{-a}^{a} \psi_1 x \psi_0\,dx\, e^{i\frac{(E_1-E_0)}{\hbar}t}\right\}$$

$$= \frac{\sqrt{3}}{2}\cos\left(\frac{(E_1-E_0)t}{\hbar}\right)\int_{-a}^{a}\psi_1 x \psi_0\,dx = \sqrt{3}\cos\left(\frac{(E_1-E_0)t}{\hbar}\right)\int_{0}^{a}\psi_1 x \psi_0\,dx$$

$$= \frac{\sqrt{3}}{a}\cos\left(\frac{(E_1-E_0)t}{\hbar}\right)\int_0^a \sin\left(\frac{\pi x}{a}\right) x \cos\left(\frac{\pi x}{2a}\right) dx$$

$$= \frac{\sqrt{3}}{a}\cos\left(\frac{\Delta E t}{\hbar}\right)\int_0^{\frac{\pi}{2}} \sin(2y)\left(\frac{2a}{\pi}\right) y \cos(y)\left(\frac{2a}{\pi}\right) dy$$

$$= \frac{4\sqrt{3}a}{\pi^2}\cos\left(\frac{\Delta E t}{\hbar}\right)\int_0^{\frac{\pi}{2}} y\sin(2y)\cos(y)\,dy = \frac{2\sqrt{3}a}{\pi^2}\cos\left(\frac{\Delta E t}{\hbar}\right)\int_0^{\frac{\pi}{2}} y(\sin y + \sin 3y)\,dy$$

$$= \frac{2\sqrt{3}a}{\pi^2}\cos\left(\frac{\Delta E t}{\hbar}\right)\left[\sin y - y\cos y + \frac{\sin(3y)}{9} - \frac{y}{3}\cos(3y)\right]_{y=0}^{\frac{\pi}{2}} \quad (\text{INT } \#319)$$

$$= \frac{2\sqrt{3}a}{\pi^2}\cos\left(\frac{\Delta E t}{\hbar}\right)\left(\frac{8}{9}\right) \Rightarrow \boxed{\langle x \rangle_t = \frac{16\sqrt{3}a}{9\pi^2}\cos\left(\frac{3}{8}\,\frac{\hbar\pi^2}{ma^2}t\right)}$$

(b) For the harmonic osc., $\psi_0(x) = \sqrt{\frac{\alpha}{\sqrt{\pi}}}\exp\left\{-\frac{1}{2}\alpha^2 x^2\right\}$, with $E_0 = \frac{1}{2}\hbar\omega$

$$\psi_2(x) = \frac{1}{\sqrt{2^{\cancel{2}}\cdot 2!}}\sqrt{\frac{\alpha}{\sqrt{\pi}}}\exp\left\{-\frac{1}{2}\alpha^2 x^2\right\}\left(\overset{2}{\cancel{4}}\alpha^2 x^2 - \overset{1}{\cancel{2}}\right)\text{, with } E_2 = \frac{5}{2}\hbar\omega$$

$\Rightarrow \psi(x,t=0) = a_0\psi_0 + a_2\psi_2$, where:

$$\sqrt{\frac{4}{3}}\,(\alpha^2 x^2) = a_0 + a_2\left[\frac{2\alpha^2x^2 - 1}{\sqrt{2}}\right]$$

$\Rightarrow \sqrt{\frac{4}{3}} = \sqrt{2}\,a_2 \Rightarrow a_2 = \sqrt{\frac{2}{3}} \Rightarrow a_1 = \sqrt{\frac{1}{3}}$, and:

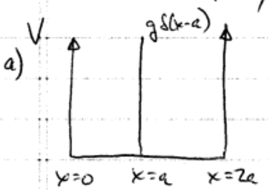
$$\psi(x,t) = \sqrt{\frac{1}{3}}\,\psi_0(x)e^{-i\frac{1}{2}\omega t} + \sqrt{\frac{2}{3}}\,\psi_2(x)e^{-i\frac{5}{2}\omega t}$$

$|\psi(x,t)|^2$ is an even fn of $x \Rightarrow \boxed{\langle x \rangle_t = 0}$ ind of time

$$\langle E \rangle_t = \frac{1}{3}E_0 + \frac{2}{3}E_2 = \frac{1}{3}\left(\frac{1}{2}\hbar\omega\right) + \frac{2}{3}\left(\frac{5}{2}\hbar\omega\right) = \boxed{\frac{11}{6}\hbar\omega\text{, ind of time}}$$

Mid-Term Exam, Problem 4

a)



$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + g\,\delta(x-a)\psi = E\psi \text{, with } \psi = 0 \text{ at } x = 0, 2a$$

ψ = cont at $x=a$

ψ is either even or odd about $x=a$

For $x \neq a$, $\frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0 \Rightarrow \psi = A\sin(kx)$ $0<x<a$

$\psi = B\sin(k(2a-x))$ $a<x<2a$

where $E = \frac{\hbar^2 k^2}{2m}$

If ψ = even, $B=A$, and cont @ $x=a$ is guaranteed

If ψ = odd, $B=-A$, and $\psi(a)=0 \Rightarrow ka = n\pi \Rightarrow k = \frac{n\pi}{a}$

$\Rightarrow E_{odd} = \frac{\hbar^2\pi^2 n^2}{2ma^2}$ are only options

Boundary cond @ $x=a$:

$$0 = \int_{a-\epsilon}^{a+\epsilon}\left\{\frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi - \frac{2mg}{\hbar^2}\delta(x-a)\psi\right\}dx$$

$$= \frac{d\psi}{dx}\Big|_{a+\epsilon} - \frac{d\psi}{dx}\Big|_{a-\epsilon} - \frac{2mg}{\hbar^2}\psi(a) \Rightarrow \Delta\left(\frac{d\psi}{dx}\right)_{x=a} = \frac{2mg}{\hbar^2}\psi(a)$$

$\frac{d\psi}{dx}\Big|_{a^+} = -Bk\cos(ka)$

$\frac{d\psi}{dx}\Big|_{a^-} = Ak\cos(ka)$

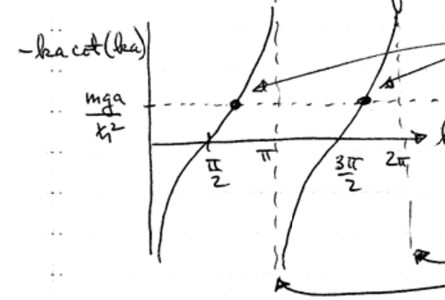
Even case $(B=A)$: $-2\cancel{A}k\cos(ka) = \frac{2mg}{\hbar^2}\cancel{A}\sin(ka)$

$$\Rightarrow \boxed{-ka\cot(ka) = \frac{mga}{\hbar^2}}$$

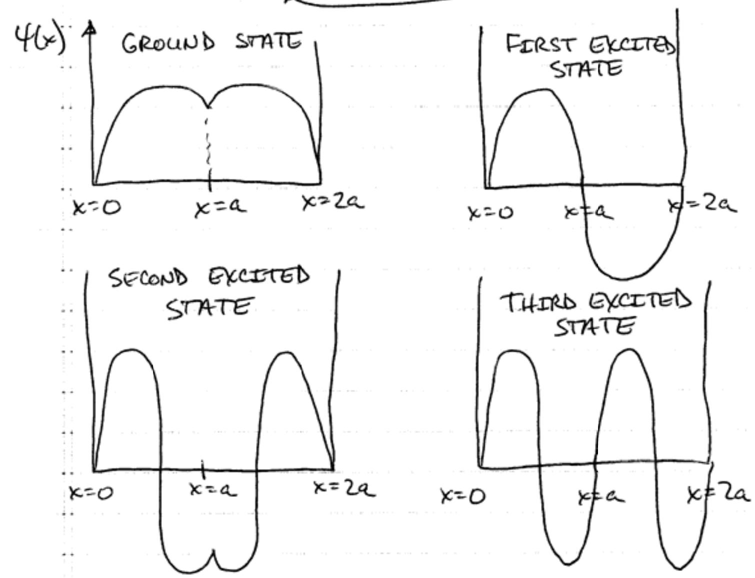
Odd case: $B=-A$: $-Ak\cos(ka) - Ak\cos(ka) = 0 = \frac{2mg}{\hbar^2}0$ ✓

(All odd "options" okay!)

b) Even solutions are roots of $-ka\cot(ka) = \frac{mga}{\hbar^2}$. LHS looks like



Odd solutions are unmodified square well eigenfns with $ka = n\pi$



(c) From the figure in part (b), when $\frac{mga}{\hbar^2}$ is very small,

$ka = \frac{\pi}{2} + \delta$, with δ small

$$\Rightarrow -\left(\frac{\pi}{2}+\delta\right)\frac{\cos\left(\frac{\pi}{2}+\delta\right)}{\sin\left(\frac{\pi}{2}+\delta\right)} = \frac{mga}{\hbar^2}$$

$$\Rightarrow \frac{mga}{\hbar^2} = +\left(\frac{\pi}{2}+\delta\right)\frac{+\sin\delta}{\cos\delta} = \left(\frac{\pi}{2}+\delta\right)\frac{\delta - \frac{1}{6}\delta^3 + \cdots}{1-\frac{1}{2}\delta^2+\cdots} \simeq \frac{\pi}{2}\delta$$

$$\Rightarrow \boxed{ka \approx \frac{\pi}{2} + \frac{2}{\pi}\frac{mga}{\hbar^2}}$$

$$\Rightarrow E_{g.s.} \approx \frac{\hbar^2 k^2}{2m} \approx \frac{\hbar^2 (ka)^2}{2ma^2} \approx \frac{\hbar^2}{2ma^2}\left[\frac{\pi^2}{4} + 2\left(\cancel{\frac{\pi}{2}}\right)\left(\cancel{\frac{2}{\pi}}\right)\frac{mga}{\hbar^2}\right]$$

$$= \frac{\hbar^2\pi^2}{8ma^2} + \frac{\cancel{\hbar^2}}{\cancel{2ma^2}}\frac{\cancel{2}\cancel{m}ga}{\cancel{\hbar^2}} = \boxed{\frac{\hbar^2\pi^2}{8ma^2} + \frac{g}{a}} = \frac{\hbar\pi^2}{8ma^2}\left[1 + \frac{8}{\pi^2}\frac{mga}{\hbar^2}\right]$$

also okay

(d) Again from part b, when $\frac{mga}{\hbar^2}$ is very large, $ka = \pi - \delta$

$$\Rightarrow \frac{mga}{\hbar^2} = -(\pi-\delta)\frac{\cos(\pi-\delta)}{\sin(\pi-\delta)} = +(\pi-\delta)\frac{+\cos\delta}{\sin\delta}$$

$$\Rightarrow \frac{\hbar^2}{mga} = \frac{1}{\pi-\delta}\frac{\sin\delta}{\cos\delta} \approx \frac{1}{\pi-\delta}\frac{\delta - \frac{1}{6}\delta^3+\cdots}{1-\frac{1}{2}\delta^2+\cdots} \simeq \frac{\delta}{\pi}$$

$$\Rightarrow ka \approx \pi\left(1 - \frac{\hbar^2}{mga}\right) \Rightarrow (ka)^2 \simeq \pi^2\left(1 - 2\frac{\hbar^2}{mga}\right), \text{ and}$$

$$E_{g.s.} = \frac{\hbar^2(ka)^2}{2ma^2} = \boxed{\frac{\hbar^2\pi^2}{2ma^2}\left(1 - 2\frac{\hbar^2}{mga}\right)}$$