Physics 606 – Final Exam

Wednesday, May 13, 2009

There are four questions on this exam. Each question is worth a total of 25 points. Please use only one side of each sheet of paper, start each problem on a new page, and encircle your final answer(s).

1. A particle with mass m is moving in one dimension under the influence of a potential of the form $V(x) = \frac{1}{2} kx^4$, where $k > 0$.

a) Find the best approximation of the ground-state energy that you can, based on a trial function of the form $\exp(-\alpha^2 x^2/2)$.

b) Describe how you could obtain a better approximation, without doing the actual calculation.

c) Describe in detail how you would obtain an approximation for the energy of the first-excited state, but don't do the calculation.

2. A *anti-commutator* of two operators F and G is given by $\{F,G\} = FG + GF$. Assume pair of dimensionless adjoint operators a and $a^\dagger$ obey the anti-commutation relation $\{a,a^\dagger\} = 1$. Let $C = a^\dagger a$, and let $C|c\rangle = c|c\rangle$ define the eigenvalues and eigenvectors of C.

a) Determine $a|c\rangle$ and $a^\dagger|c\rangle$.

b) What constraints can you place on the values of c?

c) By far the most important application of this form involves the additional conditions that $a^2 = 0$ and that the eigenvalues of C are non-degenerate, in addition to everything given above. For this special case, find explicit forms for the matrices a, $a^\dagger$, and C.

3. If the ground state of a particle of mass m is just barely bound in a spherical square well, show that the well depth V_0 and radius a are related to the binding energy E by the expression:

$$\frac{2mV_0a^2}{\hbar^2} = \frac{\pi^2}{4} + 2\kappa a + \left(1 - \frac{4}{\pi^2}\right)(\kappa a)^2 + O\left((\kappa a)^3\right)$$

where

$$\hbar\kappa = \sqrt{-2mE}$$

The deuteron provides an interesting application of this result. The p-n potential may be approximated by a square well with radius $a \approx 1.4$ fm, the Compton wavelength of the pion. The binding energy is 2.226 MeV. From these numbers, you can estimate the depth of the well. (But don't do so now, since I didn't tell you to bring calculators to this exam!)

4. In quantum mechanics, just as in classical mechanics, it's important to be able to rotate a system into an arbitrary orientation. You can do this by subjecting the quantum system to a sequence of (active) rotations about the z- and y-axes. Let's explore this process for the special case of a $j = \frac{1}{2}$ particle. Throughout this problem, assume wavefunctions (and/or kets) are column vectors expressed in terms of the normal basis where $\mathbf{J}^2$ and J_z are diagonalized simultaneously.

a) Find the matrix $U_z(\varphi)$ that produces an active rotation through an angle φ about the z-axis for an arbitrary $j = \frac{1}{2}$ vector.

b) Find the matrix $U_y(\varphi)$ that produces an active rotation through an angle φ about the y-axis for an arbitrary $j = \frac{1}{2}$ vector.

c) Let γ be an eigenvector of J_x with the maximum possible eigenvalue. Calculate $U_z\gamma$ for the special case $\varphi = \pi/2$, and show that the answer that you get makes physical sense.

d) Now calculate $U_y\gamma$ for the special case $\varphi = \pi/2$, and once again, show that the answer that you get makes physical sense.

Final Exam: Problem 1

a) $\int_{-\infty}^{+\infty} e^{-\alpha^2x^2}dx = 2\int_0^{\infty} e^{-\alpha^2x^2}dx \overset{\text{INT \#524}}{=} \cancel{2}\frac{\sqrt{\pi}}{\cancel{2}\alpha} \Rightarrow \psi = \sqrt{\frac{\alpha}{\sqrt{\pi}}}\, e^{-\frac{1}{2}\alpha^2x^2}$

is normalized.

$$\langle V\rangle = \frac{\alpha}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\left(\frac{1}{2}kx^4\right)e^{-\alpha^2x^2}dx = \frac{\alpha k}{\sqrt{\pi}}\int_0^{\infty} x^4 e^{-\alpha^2x^2}dx$$

$$\overset{\text{INT \#527}}{=} \frac{\cancel{\alpha} k}{\cancel{\sqrt{\pi}}}\,\frac{1\cdot 3}{8(\alpha^2)^2}\sqrt{\frac{\cancel{\pi}}{\cancel{\alpha^2}}} = \frac{3k}{8\alpha^4}$$

$$\langle T\rangle = \frac{\langle p\psi|p\psi\rangle}{2m} = \frac{\hbar^2\alpha}{2m\sqrt{\pi}}\int_{-\infty}^{+\infty}\left[\frac{d}{dx}\left(e^{-\frac{1}{2}\alpha^2x^2}\right)\right]^2dx = \frac{\hbar^2\alpha}{m\sqrt{\pi}}\int_0^{\infty}\alpha^4x^2e^{-\alpha^2x^2}dx$$

$$= \frac{\hbar^2}{m}\frac{\alpha}{\sqrt{\pi}}\alpha^4\int_0^{\infty}x^2e^{-\alpha^2x^2}dx \overset{\text{INT \#527}}{=} \frac{\hbar^2}{m}\frac{\cancel{\alpha}}{\cancel{\sqrt{\pi}}}\alpha^4\frac{1}{4\cancel{\alpha^2}}\sqrt{\frac{\cancel{\pi}}{\cancel{\alpha^2}}} = \frac{\hbar^2\alpha^2}{4m}$$

(Note: $\langle\psi|p^2|\psi\rangle$ gives same result.)

$$\Rightarrow \langle H(\alpha)\rangle = \frac{\hbar^2\alpha^2}{4m} + \frac{3k}{8\alpha^4}. \quad \frac{dH}{d\alpha} = \frac{\hbar^2\cancel{2}\alpha}{\cancel{4}_2 m} - \frac{\cancel{4}\cdot 3k}{\cancel{8}_2\alpha^5} = 0$$

$$\Rightarrow \frac{\hbar^2\alpha}{\cancel{2}m} = \frac{3k}{\cancel{2}\alpha^5} \Rightarrow \alpha^6 = \frac{3km}{\hbar^2} \Rightarrow \alpha^2 = \left(\frac{3km}{\hbar^2}\right)^{1/3}$$

$\Rightarrow$ The best approximation is: $\frac{\hbar^2}{4m}\left(\frac{3km}{\hbar^2}\right)^{1/3} + \frac{3k}{8}\left(\frac{\hbar^2}{3km}\right)^{2/3} = \frac{3}{8}\left(\frac{3k\hbar^4}{m^2}\right)^{1/3}$

b) Some possible improvements:

- $\psi \to \psi f_{even}(x)$, where f_{even} is an even fn of x like $a+bx^2(+cx^4+\ldots)$
- Improve the asymptotic behavior: $e^{-\frac{1}{3}\alpha^3|x|^3}$ would be better

c) Substitute $\psi \to \psi f_{odd}(x)$, where f_{odd} is an odd fn of x like ax, or $ax+bx^3$, or ...

Final Exam: Problem 2

a) $\{C,a\} = a^+aa + aa^+a = a^+aa + (1-a^+a)a = a$

$\{C,a^+\} = a^+aa^+ + a^+a^+a = a^+(1-a^+a) + a^+a^+a = a^+$

Thus, $a|c\rangle = \{C,a\}|c\rangle = Ca|c\rangle + aC|c\rangle = C(a|c\rangle) + c(a|c\rangle)$

$\Rightarrow C(a|c\rangle) = (1-c)(a|c\rangle)$, so $a|c\rangle$ is an eigenvector of C with eigenvalue $1-c$. Thus, it must be proportional to $|1-c\rangle$.

To find the proportionality constant: $c = \langle c|a^+a|c\rangle = \|a|c\rangle\|^2$

$\Rightarrow \|a|c\rangle\| = \sqrt{c}$, and $\boxed{a|c\rangle = \sqrt{c}\,|1-c\rangle}$.

Similarly, $a^+|c\rangle$ is proportional to $|1-c\rangle$. To find the proportionality constant:

$\|a^+|c\rangle\|^2 = \langle a^+|c\rangle | a^+|c\rangle\rangle = \langle c|aa^+|c\rangle = \langle c|1-a^+a|c\rangle$

$= 1-c \Rightarrow \boxed{a^+|c\rangle = \sqrt{1-c}\,|1-c\rangle}$.

b) $\|a|c\rangle\|^2 \geq 0 \Rightarrow c \geq 0$. $\|a^+|c\rangle\|^2 \geq 0 \Rightarrow 1-c \geq 0 \Rightarrow 1 \geq c \Rightarrow$

$\boxed{0 \leq c \leq 1}$. Note: c is a dimensionless number.

c) $C^2|c\rangle = c^2|c\rangle = (a^+a)(a^+a)|c\rangle = a^+(1-a^+a)a|c\rangle = (a^+a - (a^+)^2(a^2))|c\rangle$

$= C|c\rangle = c|c\rangle$

$\Rightarrow c^2 = c \Rightarrow \boxed{c = 0 \text{ or } 1}$. From the forms for a + a^+, both must

FEx.2.2

exist $\Rightarrow$ $\boxed{C = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}$ $\left(\underline{\underline{\text{Note}}} : \Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle \; ; \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle \right)$

$a|0\rangle = 0 \; ; \; a|1\rangle = |0\rangle \Rightarrow$ $\boxed{a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}$

$\Rightarrow$ $\boxed{a^{+} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}$

which is consistent with $a^{+}|0\rangle = |1\rangle$, $a^{+}|1\rangle = 0$. ✓

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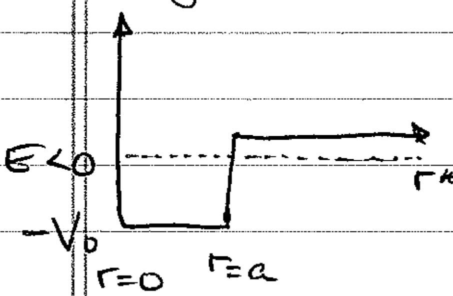
Note : a and a^{+} obey the algebra for annihilation and creation operators for fermions.

FEx. 3

Final Exam: Problem 3

The ground state must have $l=0 \Rightarrow -\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + V(r)u = Eu$

with $u(0)=0$ and $u(r\to\infty)\to 0$



$$\Rightarrow \frac{d^2u}{dr^2} + \frac{2m}{\hbar^2}\left(E - V(r)\right)u = 0$$

$r<a$: $u(r) = A\sin(kr)$, with $k = \sqrt{\frac{2m(V_0 - |E|)}{\hbar^2}}$

$r>a$: $u(r) = Be^{-Kr}$, with $K = \sqrt{\frac{2m|E|}{\hbar^2}}$

Continuity @ $r=a \Rightarrow$ $A\sin(ka) = Be^{-Ka}$

$Ak\cos(ka) = -KBe^{-Ka}$

$\Rightarrow k\cot(ka) = -K$. Mult by a for dimensionless $\Rightarrow$

$\boxed{-ka\cot(ka) = Ka}$ $\Rightarrow$ $\cot(ka)$ must be negative $\Rightarrow$

$ka = \frac{\pi}{2} + \delta$ for just barely bound.

Meanwhile, $(ka)^2 + (Ka)^2 = \frac{2m(V_0 - |E|)a^2}{\hbar^2} + \frac{2m|E|a^2}{\hbar^2} = \frac{2mV_0a^2}{\hbar^2}$

$$\Rightarrow \frac{2mV_0a^2}{\hbar^2} = \left(\frac{\pi}{2}+\delta\right)^2 + (Ka)^2 = \frac{\pi^2}{4} + \pi\delta + \delta^2 + (Ka)^2$$

$$\cot(ka) = \cot\left(\frac{\pi}{2}+\delta\right) = \frac{\cos\left(\frac{\pi}{2}+\delta\right)}{\sin\left(\frac{\pi}{2}+\delta\right)} = \frac{-\sin\delta}{\cos\delta} \approx \frac{-\delta}{1-\frac{1}{2}\delta^2} \Rightarrow$$

$$\left(\frac{\pi}{2}+\delta\right)\frac{\delta}{1-\frac{1}{2}\delta^2} \approx \frac{\pi}{2}\delta + \delta^2 \approx Ka \Rightarrow 2Ka \approx \pi\delta + 2\delta^2 + O(\delta^3)$$

$$\Rightarrow \frac{2mV_0a^2}{\hbar^2} = \frac{\pi^2}{4} + 2Ka + (Ka)^2 - \delta^2 = \boxed{\frac{\pi^2}{4} + 2Ka + \left(1 - \frac{4}{\pi^2}\right)(Ka)^2 + O\left((Ka)^3\right)}$$

Final Exam: Problem 4

a) $J_z = \begin{pmatrix} +\frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{pmatrix} \Rightarrow U_z(\phi) = e^{-\frac{i}{\hbar}\phi J_z} = \begin{pmatrix} e^{-\frac{i}{\hbar}\phi\frac{\hbar}{2}} & 0 \\ 0 & e^{-\frac{i}{\hbar}\phi(-\frac{\hbar}{2})} \end{pmatrix}$

$$= \begin{pmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{+i\frac{\phi}{2}} \end{pmatrix}$$

b) $J_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\quad U_y(\phi) = e^{-\frac{i}{\hbar}\phi J_y}$. To calculate this, first diagonalize J_y:

$\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \pm\frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \pm ia = b \Rightarrow$ eigenvectors are:

$+\frac{\hbar}{2}: \begin{pmatrix} a \\ ia \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$ (normalized) $\quad -\frac{\hbar}{2}: \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}$

Let $S = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \quad S^+ = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix}$

Then $S J_y S^+ = \frac{\hbar}{2}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix}$

$$= \frac{\hbar}{2}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{pmatrix}$$

$$U_y(\phi) = S^+ S U_y(\phi) S^+ S = S^+ e^{-\frac{i}{\hbar}\phi S J_y S^+} S = S^+\begin{pmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{\frac{i\phi}{2}} \end{pmatrix} S$$

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$$\Rightarrow U_y(\phi) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{i\frac{\phi}{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{e^{-i\frac{\phi}{2}}}{\sqrt{2}} & \frac{e^{+i\frac{\phi}{2}}}{\sqrt{2}} \\ \frac{ie^{-i\frac{\phi}{2}}}{\sqrt{2}} & \frac{-ie^{+i\frac{\phi}{2}}}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{e^{i\frac{\phi}{2}}+e^{-i\frac{\phi}{2}}}{2} & \frac{ie^{i\frac{\phi}{2}}-ie^{-i\frac{\phi}{2}}}{2} \\ \frac{-ie^{i\frac{\phi}{2}}+ie^{-i\frac{\phi}{2}}}{2} & \frac{e^{i\frac{\phi}{2}}+e^{-i\frac{\phi}{2}}}{2} \end{pmatrix}$$

$$\Rightarrow \boxed{U_y(\phi) = \begin{pmatrix} \cos\left(\frac{\phi}{2}\right) & -\sin\left(\frac{\phi}{2}\right) \\ \sin\left(\frac{\phi}{2}\right) & \cos\left(\frac{\phi}{2}\right) \end{pmatrix}}$$

c) $J_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \gamma = \begin{pmatrix} a \\ b \end{pmatrix}$ with $\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix}$

$\Rightarrow b = a \Rightarrow \gamma = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

Then $U_z\left(\frac{\pi}{2}\right)\gamma = \begin{pmatrix} e^{-i\frac{\pi}{4}} & 0 \\ 0 & e^{+i\frac{\pi}{4}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2}} \\ \frac{e^{+i\frac{\pi}{4}}}{\sqrt{2}} \end{pmatrix}$

$\Rightarrow \boxed{U_z\left(\frac{\pi}{2}\right)\gamma = e^{-i\frac{\pi}{4}}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}}$ is an eigenvector of J_y with eigenvalue $+\frac{\hbar}{2}$.

A 90° rotation about z takes +x axis to the +y axis, so this makes sense. ✓

FEx.4.3

d) $\mathcal{U}_y\left(\frac{\pi}{2}\right)\gamma = \begin{pmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}-\frac{1}{2} \\ \frac{1}{2}+\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

This is an eigenvector of J_z with eigenvalue $-\frac{\hbar}{2}$.

A $90°$ rotation about the y axis takes the +x axis to the -z axis, so this also makes sense.