PHYS 606 – Spring 2017 – Homework I

Problem [1]

(a) Hamilton fct. $H = \frac{p^2}{2m} + \frac{k}{2}x^2 = E \Rightarrow \frac{p^2}{2mE} + \frac{x^2}{\frac{2E}{k}} = 1$

$\Rightarrow$ Phase space motion is ellipse with semi axes $\sqrt{2mE}$ and $\sqrt{\frac{2E}{k}}$

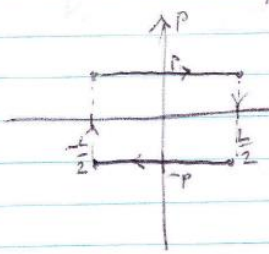
Bohr – Sommerfeld: $\oint p\,dq = \underbrace{\pi\sqrt{2mE}\sqrt{\frac{2E}{k}}}_{\text{area of ellipse}} = 2\pi\frac{E}{\omega}$ with $\omega = \sqrt{\frac{k}{m}}$

On the other hand $\oint p\,dq \doteq nh$

$\Rightarrow E = n\,\frac{h\omega}{2\pi} = n\hbar\omega$

All these energy levels are off by $\frac{1}{2}\hbar\omega$ from the full QM result, but the radiation spectrum (involving ΔE) can be predicted accurately.

(b) Consider particle with momentum p (moving right); reflected at $x = +\frac{L}{2}$ to obtain momentum $-p$ (energy conserved); another reflection $-p \to p$ at $x = -\frac{L}{2}$



$\oint p\,dq = 2pL \doteq nh$

$\Rightarrow E = \frac{p^2}{2m} = n^2\frac{h^2}{8mL^2}$

same as the full QM result!

Problem [2]

(a) $$|C|^2 \int_{\mathbb{R}} e^{-\frac{(x-x_0)^2}{2\sigma^2}}\,dx \underset{z=\frac{x-x_0}{\sqrt{2}\sigma}}{=} |C|^2 \int_{\mathbb{R}} e^{-z^2}\,dz \cdot \sqrt{2}\,\sigma = |C|^2 \sqrt{2\pi}\,\sigma$$

[One bonus point if you calculated that integral yourself:

$$\int_{\mathbb{R}} e^{-z^2}\,dz = \left(\int_{\mathbb{R}^2} e^{-r^2}\,d^2r\right)^{1/2} = \left(2\pi \int_0^\infty r e^{-r^2}\,dr\right)^{1/2} \underset{u=r^2}{=} \left(\pi \int_0^\infty e^{-u}\,du\right)^{1/2} = \sqrt{\pi}$$]

$$\Rightarrow C = \frac{1}{\sqrt{\sqrt{2\pi}\,\sigma}} \qquad (\times \text{ a phase which we ignore here})$$

(b) $$\langle x\rangle = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{\mathbb{R}} x\, e^{-\frac{(x-x_0)^2}{2\sigma^2}}\,dx \underset{u=\frac{x-x_0}{\sqrt{2}\sigma}}{=} \frac{1}{\sqrt{2\pi}\,\sigma}\int_{\mathbb{R}} \left(\sqrt{2}\sigma u + x_0\right) e^{-u^2}\sqrt{2}\,\sigma\,du$$

$$= \frac{x_0}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-u^2}\,du = x_0$$

$\int u e^{-u^2}du = 0$

$$(\Delta x)^2 = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{\mathbb{R}} (x-x_0)^2 e^{-\frac{(x-x_0)^2}{2\sigma^2}}\,dx \underset{u=\frac{x-x_0}{\sqrt{2}\sigma}}{=} \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} 2\sigma^2 u^2 e^{-u^2}\,du = -\frac{\sigma^2}{\sqrt{\pi}}\int_{\mathbb{R}} u \frac{d}{du} e^{-u^2}\,du$$

$$\underset{\text{part. int}}{=} \underbrace{\left[-\frac{\sigma^2}{\sqrt{\pi}} u e^{-u^2}\right]_{-\infty}^{+\infty}}_{0} + \frac{\sigma^2}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-u^2}\,du = \sigma^2$$

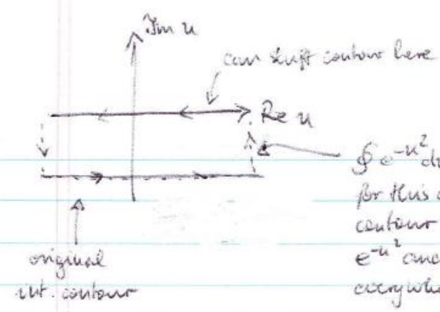
Center x_0 and width σ are equal to "average x" and $\sqrt{\text{variance}} = \Delta x$, resp.

(c) $$\hat{f}(k) = (2\pi)^{-1/2}\frac{1}{\sqrt{\sqrt{2\pi}\,\sigma}}\int_{\mathbb{R}} e^{-\frac{(x-x_0)^2}{4\sigma^2}} e^{-ikx}\,dx$$

$$= \frac{1}{(2\pi)^{3/4}\sigma^{1/2}} e^{-ikx_0}\int_{\mathbb{R}} e^{-\frac{x^2}{4\sigma^2}} e^{-ikx}\,dx$$

Complete square → $$= \frac{1}{(2\pi)^{3/4}\sigma^{1/2}} e^{-ikx_0}\int_{\mathbb{R}} e^{-\left(\frac{x}{2\sigma}+ik\sigma\right)^2}\,dx\; e^{-\sigma^2k^2}$$

$$\underset{u=\frac{x}{2\sigma}+ik\sigma}{=} \frac{2^{1/4}\sigma^{1/2}}{\pi^{3/4}} e^{-ikx_0} e^{-\sigma^2k^2}\int_{-\infty-2ik\sigma^2}^{+\infty-2ik\sigma^2} e^{-u^2}\,du = \sqrt[4]{\frac{2}{\pi}}\,\sqrt{\sigma}\, e^{-ikx_0} e^{-\sigma^2k^2}$$



$\oint e^{-u^2} du = 0$ for this closed contour since e^{-u^2} analytic everywhere

so $\displaystyle\int_{-\infty-2ik\sigma^2}^{+\infty-2ik\sigma^2} e^{-u^2} du = \int_{-\infty}^{+\infty} e^{-u^2} du = \sqrt{\pi}$

We can writ this in "standard form" with a width in momentum space

$\hat{\sigma} = \frac{1}{2\sigma}$; then $\displaystyle\hat{f}(k) = \frac{1}{\sqrt{\sqrt{2\pi}\,\hat{\sigma}}} e^{-ikx_0} e^{-\frac{k^2}{4\hat{\sigma}^2}}$

[Summary: Gauss centered around 0, width σ $\xleftrightarrow{FT}$ Gauss centered around 0, width $\frac{1}{2\sigma}$

Displacement by x_0 $\xleftrightarrow{FT}$ phase factor e^{-ikx_0}]

Using the result from (b): $\langle k \rangle = 0$

(phase e^{-ikx_0} drops out) $(\Delta k)^2 = \langle k^2 \rangle = \sigma_k^2 = \frac{1}{4\sigma^2}$

(e) $\Rightarrow \Delta x \, \Delta k = \sigma \sigma_k = \frac{1}{2}$

(d) Now additional factor ~~for~~ $e^{ik_0 x}$ [note: C, $\langle x \rangle$ and $\langle \Delta x \rangle^2$ are unchanged]

Fourier integral changes to $\displaystyle\hat{f}_{k_0}(k) = \frac{1}{(2\pi)^{1/2}} \frac{1}{\sqrt{\sqrt{2\pi}\,\sigma}} \int e^{-\frac{(x-x_0)^2}{4\sigma^2}} e^{-i(k-k_0)x} dx$

$\Rightarrow$ simple shift $k \mapsto k - k_0$: $\displaystyle\hat{f}_{k_0}(k) = \hat{f}(k-k_0) = \frac{1}{\sqrt{\sqrt{2\pi}\,\hat{\sigma}}} e^{-i(k-k_0)x_0} e^{-\frac{(k-k_0)^2}{4\hat{\sigma}^2}}$

Obviously $\langle k \rangle_{k_0} = k_0$, $(\Delta k)^2_{k_0} = \langle (k-k_0)^2 \rangle = \frac{1}{4\sigma^2}$ as before

Problem [3]

Hamilton fct. $H(x,p) = \frac{p^2}{2m} - bx$

Hamilton – Jacobi: $\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 - bx + \frac{\partial S}{\partial t} = 0$ with $p = \frac{\partial S}{\partial x}$

Since $\frac{\partial H}{\partial t} = 0$ time is separable: $S = W(x) - Et$

$\Rightarrow \frac{1}{2m}\left(\frac{dW}{dx}\right)^2 = E + bx \Rightarrow \frac{dW}{dx} = \pm\sqrt{2m(E+bx)}$

$\Rightarrow W(x) = \pm \frac{1}{3mb}\left[2m(E+bx)\right]^{3/2} + \text{const.}$

$\Rightarrow S(x,t) = \pm \frac{1}{3mb}\left[2m(E+bx)\right]^{3/2} - Et + \text{const.}$

E is constant of motion choose it a the variable after canonical transf.

$\Rightarrow$ Associated momentum $\beta = \frac{\partial S}{\partial \alpha} = \text{const.}$ and $\beta = \pm\frac{1}{b}\sqrt{2m(E+bx)} - t$

$\Rightarrow x = \frac{b}{2m}(t+\beta)^2 - \frac{E}{b}$

Initial conditions: $x(0) = \frac{b\beta^2}{2m} - \frac{E}{b} \overset{!}{=} x_0$ $\quad \dot{x}(0) = \frac{b\beta}{m} \overset{!}{=} v_0$

$\Rightarrow \beta = \frac{m}{b}v_0$ and $E = \frac{b^2}{2m}\frac{m^2}{b^2}v_0^2 - bx_0 = \frac{1}{2}mv_0^2 - bx_0$

$\Rightarrow x(t) = \frac{b}{2m}\left(t + \frac{m}{b}v_0\right)^2 - \frac{mv_0^2}{2b} + x_0$