

PHYS 606 – Spring 2016 – Homework I

Problem [1]

(a) Hamilton pt. $H = \frac{p^2}{2m} + \frac{k}{2}x^2 = E \Rightarrow \frac{p^2}{2mE} + \frac{x^2}{\frac{2E}{k}} = 1$

\Rightarrow Phasespace motion is ellipse with semi axes $\sqrt{2mE}$ and $\sqrt{\frac{2E}{k}}$

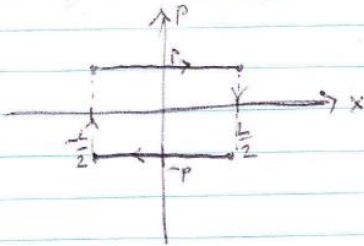
Bohr-Sommerfeld: $\oint p dq = \pi \sqrt{2mE} \sqrt{\frac{2E}{k}} = 2\pi \frac{E}{\omega}$ with $\omega = \sqrt{\frac{k}{m}}$
area of ellipse

On the other hand $\oint p dq \stackrel{!}{=} nh$

$\Rightarrow E = n \frac{\hbar\omega}{2\pi} = n\hbar\omega$

All these energy levels are off by $\frac{1}{2}\hbar\omega$ from the full QM result, but the radiation spectrum (involving ΔE) can be predicted accurately.

(b) Consider particle with momentum p (moving right); reflected at $x = +\frac{L}{2}$ to obtain momentum $-p$ (energy conserved); another reflection $-p \rightarrow p$ at $x = -\frac{L}{2}$



$\oint p dq = 2pL \stackrel{!}{=} nh$

$\Rightarrow E = \frac{p^2}{2m} = n^2 \frac{\hbar^2}{8mL^2}$

same as the full QM result!

Problem [2]

$$(a) \quad |C|^2 \int_{\mathbb{R}} e^{-\frac{(x-x_0)^2}{2\sigma^2}} dx = |C|^2 \int_{\mathbb{R}} e^{-z^2} dz \cdot \sqrt{2}\sigma = |C|^2 \sqrt{2\pi} \sigma$$

$z = \frac{x-x_0}{\sqrt{2}\sigma}$

One bonus point if you calculated that integral yourself:

$$\int_{\mathbb{R}} e^{-z^2} dz = \left(\int_{\mathbb{R}^2} e^{-r^2} d^2r \right)^{1/2} = \left(2\pi \int_0^\infty r e^{-r^2} dr \right)^{1/2} = \left(\pi \int_0^\infty e^{-u} du \right)^{1/2} = \sqrt{\pi}$$

$u=r^2$

$$\Rightarrow C = \frac{1}{\sqrt{2\pi}\sigma} \quad (\text{x phase which we ignore here})$$

$$(b) \quad \langle x \rangle = \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} x e^{-\frac{(x-x_0)^2}{2\sigma^2}} dx = \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} (\sqrt{2}\sigma u + x_0) e^{-u^2} \sqrt{2}\sigma du$$

$u = \frac{x-x_0}{\sqrt{2}\sigma}$

$$= \frac{x_0}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-u^2} du = x_0$$

$\int_{\mathbb{R}} u e^{-u^2} du = 0$

$$\langle \Delta x \rangle^2 = \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} (x-x_0)^2 e^{-\frac{(x-x_0)^2}{2\sigma^2}} dx = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} 2\sigma^2 u^2 e^{-u^2} du = -\frac{\sigma^2}{\sqrt{\pi}} \int_{\mathbb{R}} u \frac{d}{du} e^{-u^2} du$$

$u = \frac{x-x_0}{\sqrt{2}\sigma}$

$$= \left[-\frac{\sigma^2}{\sqrt{\pi}} u e^{-u^2} \right]_{-\infty}^{+\infty} + \frac{\sigma^2}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-u^2} du = \sigma^2$$

part. int.

Center x_0 and width σ are equal to "average x " and $\sqrt{\text{variance}} = \Delta x$, resp.

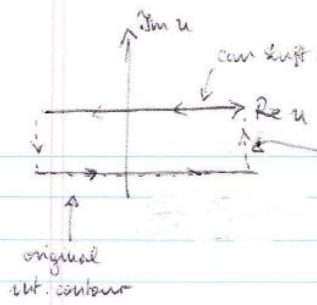
$$(c) \quad \hat{f}(k) = (2\pi)^{-1/2} \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} e^{-\frac{(x-x_0)^2}{4\sigma^2}} e^{-ikx} dx$$

$$= \frac{1}{(2\pi)^{3/4} \sigma^{1/2}} e^{-ikx_0} \int_{\mathbb{R}} e^{-\frac{x^2}{4\sigma^2}} e^{-ikx} dx$$

Complete square \rightarrow
$$= \frac{1}{(2\pi)^{3/4} \sigma^{1/2}} e^{-ikx_0} \int_{\mathbb{R}} e^{-\left(\frac{x}{2\sigma} + ik\sigma\right)^2} dx e^{-\sigma^2 k^2}$$

$$= \frac{2^{1/4} \sigma^{1/2}}{\pi^{3/4}} e^{-ikx_0} e^{-\sigma^2 k^2} \int_{-\infty - 2ik\sigma^2}^{+\infty - 2ik\sigma^2} e^{-u^2} du = \frac{\sqrt{2}}{\sqrt{\pi}} \sqrt{\sigma} e^{-ikx_0} e^{-\sigma^2 k^2}$$

$u = \frac{x}{2\sigma} + ik\sigma$



$$\oint e^{-u^2} du = 0$$
 so
$$\int_{-x-2ik\sigma^2}^{+x-2ik\sigma^2} e^{-u^2} du = \int_{-x}^{+x} e^{-u^2} du = \sqrt{\pi}$$

We can write this in "standard form" with a width in momentum space

$\sigma = \frac{1}{2\sigma}$; then
$$\hat{f}(k) = \frac{1}{\sqrt{2\pi} \sigma} e^{-ikx_0} e^{-\frac{k^2}{4\sigma^2}}$$

[Summary:	Gauss centered around 0, width σ	$\xleftrightarrow{\text{FT}}$	Gauss centered around 0, width $\frac{1}{2\sigma}$
		Displacement by x_0	$\xleftrightarrow{\text{FT}}$	phase factor e^{-ikx_0}
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Using the result from (b): $\langle k \rangle = 0$

(phase e^{-ikx_0} drops out)

$$(\Delta k)^2 = \langle k^2 \rangle = \sigma_k^2 = \frac{1}{4\sigma^2}$$

$$\Rightarrow \Delta x \Delta k = \sigma \sigma_k = \frac{1}{2}$$

Problem [3]

$$\text{Hamilton fct. } H(x, p) = \frac{p^2}{2m} - bx$$

$$\text{Hamilton-Jacobi: } \frac{1}{2m} \left(\frac{\partial S}{\partial x} \right)^2 - bx + \frac{\partial S}{\partial t} = 0 \quad \text{with } p = \frac{\partial S}{\partial x}$$

$$\text{Since } \frac{\partial H}{\partial t} = 0 \text{ time is separable: } S = W(x) - Et$$

$$\Rightarrow \frac{1}{2m} \left(\frac{dW}{dx} \right)^2 = E + bx \Rightarrow \frac{dW}{dx} = \pm \sqrt{2m(E + bx)}$$

$$\Rightarrow W(x) = \pm \frac{1}{3mb} [2m(E + bx)]^{3/2} + \text{const.}$$

$$\Rightarrow S(x, t) = \pm \frac{1}{3mb} [2m(E + bx)]^{3/2} - Et + \text{const.}$$

E is constant of motion choose it a the variable after canonical transf.

$$\Rightarrow \text{Associated momentum } \beta = \frac{\partial S}{\partial x} = \text{const. and } \beta = \pm \frac{1}{b} \sqrt{2m(E + bx)} - L$$

$$\Rightarrow x = \frac{b}{2m} (t + \beta)^2 - \frac{E}{b}$$

$$\text{Initial conditions: } x(0) = \frac{b\beta^2}{2m} - \frac{E}{b} \stackrel{!}{=} x_0 \quad \dot{x}(0) = \frac{b\beta}{m} \stackrel{!}{=} v_0$$

$$\Rightarrow \beta = \frac{m}{b} v_0 \text{ and } E = \frac{b^2}{2m} \frac{m^2}{b^2} v_0^2 - bx_0 = \frac{1}{2} m v_0^2 - bx_0$$

$$\Rightarrow x(t) = \frac{b}{2m} \left(t + \frac{m}{b} v_0 \right)^2 - \frac{m v_0^2}{2b} + x_0$$