PHYS 606 – Spring 2015 – Homework VI – Solution

Problem [1]

(a) Use power series

$$\frac{dH_n}{d\xi} \underset{n\text{ even}}{=} \frac{d}{d\xi}\left[(-1)^{\frac{n}{2}}\frac{n!}{(\frac{n}{2})!}\left(1-\frac{2n}{2!}\xi^2+\frac{2^2n(n-2)}{4!}-\dots\right)\right]$$

$$= (-1)^{\frac{n}{2}}\frac{n!}{(\frac{n}{2})!}\left(-\frac{2n}{1!}\xi+\frac{2^2n(n-2)}{3!}\xi^3-\dots\right)$$

$$= -2n(-1)^{\frac{n}{2}}\frac{n}{\frac{n}{2}}\frac{(n-1)!}{(\frac{n}{2}-1)!}\left(\xi-\frac{2(n-2)}{3!}\xi^3+\dots\right)$$

$$= 2n(-1)^{\frac{n-2}{2}}\frac{2(n-1)!}{(\frac{n-2}{2})!}\left(\xi-\frac{2(n-2)}{3!}\xi^3+\dots\right) = 2n\, \underset{(n-1)\text{ odd}}{H_{n-1}(\xi)}$$

$$\frac{dH_n}{d\xi} \underset{n\text{ odd}}{=} \frac{d}{d\xi}\left[(-1)^{\frac{n-1}{2}}\frac{2n!}{(\frac{n-1}{2})!}\left(\xi-\frac{2(n-1)}{3!}\xi^3+\frac{2^2(n-1)(n-3)}{5!}\xi^5-\dots\right)\right]$$

$$= 2n(-1)^{\frac{n-1}{2}}\frac{(n-1)!}{(\frac{n-1}{2})!}\left(1-\frac{2(n-1)}{2!}\xi^2+\frac{2^2(n-1)(n-3)}{4!}\xi^4-\dots\right)$$

$$= 2n\, \underset{(n-1)\text{ even}}{H_{n-1}(\xi)}$$

(b) $$\frac{d}{d\xi}F(\xi,s) = \sum_{n=0}^{\infty}\frac{dH_n}{d\xi}\frac{s^n}{n!} = 2\sum_{n=1}^{\infty}\frac{H_{n-1}}{(n-1)!}s^n = 2s\sum_{n=0}^{\infty}\frac{H_n}{n!}s^n = 2sF(\xi,s)$$

$$\Rightarrow F(\xi,s) = C'e^{2\xi s}$$

and $$F(0,s) = \sum_{n=0}^{\infty}\frac{H_n(0)}{n!}s^n \underset{\substack{\text{only constant}\\\text{term of even } n \text{ contribute}}}{=} \sum_{\substack{n=0\\n\text{ even}}}^{\infty}(-1)^{\frac{n}{2}}\frac{n!}{(\frac{n}{2})!}\frac{s^n}{n!} \underset{n\mapsto 2n}{=} \sum_{n=0}^{\infty}(-1)^n\frac{s^{2n}}{n!} = e^{-s^2} \equiv C'$$

$$\Rightarrow F(\xi,s) = e^{-s^2+2\xi s} = e^{\xi^2-(s-\xi)^2}$$

(c) $$H_n(\xi) = \frac{d^n}{ds^n}F(\xi,s)\Big|_{s=0} = e^{\xi^2}\frac{d^n}{ds^n}e^{-(s-\xi)^2}\Big|_{s=0} = (-1)^n e^{\xi^2}\frac{d^n}{d\xi^n}e^{-(s-\xi)^2}\Big|_{s=0} = (-1)e^{\xi^2}\frac{d^n}{d\xi^n}e^{-\xi^2}$$

(d) Induction: $$\frac{d}{d\xi}\frac{2^n}{\sqrt{\pi}}\int_{\mathbb{R}}(\xi+is)^n e^{-s^2}ds = 2n\frac{2^{n-1}}{\sqrt{\pi}}\int_{\mathbb{R}}(\xi+is)^{n-1}e^{-s^2}ds$$

same as the $H_n(\xi)$, see (a); and

$$H_0(\xi) = \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}1\cdot e^{-s^2}ds = 1 \quad ✓$$

Problem [2]

(a) $I = \int_{\mathbb{R}} F(\xi, s)\, F(\xi, t)\, e^{-\xi^2 + 2\lambda\xi}\, d\xi =$

$\overset{(a)}{=} \int_{\mathbb{R}} e^{\xi^2 - (s-\xi)^2 + \xi^2 - (t-\xi)^2}\, e^{-\xi^2 + 2\lambda\xi}\, d\xi$

$= e^{\lambda^2 + 2(st + \lambda s + \lambda t)} \int_{\mathbb{R}} e^{-(\xi - s - t - \lambda)^2}\, d\xi = \sqrt{\pi}\, e^{\lambda^2 + 2(st + \lambda s + \lambda t)}$

On the other hand $\bar{I} = \sum_{n,m,k=0}^{\infty} \int_{\mathbb{R}} H_n(\xi) \frac{s^n}{n!} H_m(\xi) \frac{t^m}{m!}\, e^{-\xi^2}\, \xi^k \frac{(2\lambda)^k}{k!}$

$\Rightarrow I_{nmk} = \int_{\mathbb{R}} H_n(\xi)\, H_m(\xi)\, e^{-\xi^2}\, \xi^k\, d\xi = \frac{1}{2^k} \frac{\partial^{n+m+k} I}{\partial s^n \partial t^m \partial \lambda^k}\Bigg|_{s,t,\lambda=0}$

(b) $\langle \psi_n | x \psi_{n'} \rangle = \int_{\mathbb{R}} \psi_n(x)\, x\, \psi_{n'}(x)\, dx = 2^{-\frac{1}{2}(n+n')} \frac{1}{\sqrt{n!\,n'!}} \sqrt{\frac{\hbar}{m\omega\pi}} \int H_n(\xi)\, H_{n'}(\xi)\, e^{-\xi^2}\, \xi\, d\xi$

$\xi = \sqrt{\frac{m\omega}{\hbar}}\, x$

$\overset{(a)}{=} 2^{-\frac{1}{2}(n+n')} \frac{1}{\sqrt{n!\,n'!}} \sqrt{\frac{\hbar}{m\omega\pi}}\, I_{nn'1}$

$\underset{(a)}{=} 2^{-\frac{1}{2}(n+n')} \frac{1}{\sqrt{n!\,n'!}} \sqrt{\frac{\hbar}{m\omega}}\, \frac{1}{2} \frac{\partial^{n+n'}}{\partial s^n \partial t^{n'}}\Bigg|_{s,t=0} \underbrace{\left[2(s+t)\, e^{2st}\right]}_{\longrightarrow\; 2\sum_{\ell=0}^{\infty}\left[\frac{s^{\ell+1}(2t)^{\ell}}{\ell!} + \frac{t^{\ell+1}(2s)^{\ell}}{\ell!}\right]}$

$= 2^{-\frac{1}{2}(n+n')} \frac{1}{\sqrt{n!\,n'!}} \sqrt{\frac{\hbar}{m\omega}} \sum_{\ell=0}^{\infty} \left[\delta_{n,\ell+1}\, \delta_{n',\ell}\, n!\, 2^{n'} + \delta_{n',\ell+1}\, \delta_{n,\ell}\, n'!\, 2^{n}\right]$

$= \sqrt{\frac{\hbar}{m\omega}} \left[\delta_{n,n'+1} \sqrt{n}\, 2^{-\frac{1}{2}} + \delta_{n',n+1} \sqrt{n'}\, 2^{-\frac{1}{2}}\right]$

$= \sqrt{\frac{\hbar}{2m\omega}} \left[\delta_{n',n-1} \sqrt{n} + \delta_{n',n+1} \sqrt{n+1}\right]$

(c) $\langle \psi_n | x^2 \psi_{n'} \rangle \underset{\text{as in (b)}}{=} 2^{-\frac{1}{2}(n+n')} \frac{1}{\sqrt{n!\,n'!}} \frac{\hbar}{m\omega} \frac{1}{\sqrt{\pi}}\, I_{nn'2}$

$= 2^{-\frac{1}{2}(n+n')} \frac{1}{\sqrt{n!\,n'!}} \frac{\hbar}{m\omega} \frac{1}{4} \frac{\partial^{n+n'}}{\partial s^n \partial t^{n'}}\Bigg|_{s,t=0} \underbrace{\left[\left(2 + 4(s+t)^2\right) e^{2st}\right]}_{2\sum_{\ell=0}^{\infty}\left[\frac{(2st)^{\ell} + 2(2st)^{\ell+1}}{\ell!} + \frac{2s^{\ell+2}(2t)^{\ell}}{\ell!} + \frac{2t^{\ell+2}(2s)^{\ell}}{\ell!}\right]}$

$$= 2^{-\frac{1}{2}(n+n')} \frac{1}{\sqrt{n!\,n'!}} \frac{\hbar}{m\omega} \frac{1}{2} \sum_{\ell=0}^{\infty} \Big[\delta_{n,\ell}\delta_{n',\ell} 2^{n} n! + \delta_{n,\ell+1}\delta_{n',\ell+1} 2^{n+1} n\, n! + \delta_{n,\ell+2}\delta_{n',\ell} 2^{n'+1} n'! + \delta_{n',\ell+2}\delta_{n,\ell} 2^{n+1} n'! \Big]$$

$$= \frac{\hbar}{2m\omega} \left[\delta_{n,n'} + \delta_{n,n'} 2n + \delta_{n,n'+2} \sqrt{n(n-1)} + \delta_{n',n+2} \sqrt{n'(n'-1)} \right]$$

$$= \frac{\hbar}{2m\omega} \left[(2n+1)\delta_{n,n'} + \delta_{n',n-2} \sqrt{n(n-1)} + \delta_{n',n+2} \sqrt{(n+2)(n-1)} \right]$$

Problem [3]

[3] Let $\psi(x,0) = \sum_{n\in\mathbb{N}} c_n \psi_n(x)$ with ψ_n energy eigenstates

$$\psi(x,t) = \sum_{n\in\mathbb{N}} c_n \psi_n(x)\, e^{-\frac{i}{\hbar}((n+\frac{1}{2})\hbar\omega)t}$$

$$\langle x \rangle = \langle \psi(x,t) | x\ \psi(x,t) \rangle = \sum_{n,n'\in\mathbb{N}} c_n^* c_{n'} e^{-i(n'-n)\omega t} \langle \psi_n | x\, \psi_{n'} \rangle$$

$$\underset{[2](b)}{=} \sum_{n,n'} c_n^* c_{n'} e^{-i(n'-n)\omega t} \sqrt{\frac{\hbar}{2m\omega}} \left[\delta_{n',n-1} \sqrt{n} + \delta_{n',n+1} \sqrt{n+1} \right]$$

$$= \sum_{n\in\mathbb{N}} \left[c_n^* c_{n-1} e^{i\omega t} \sqrt{n} + c_{n-1}^* c_n \sqrt{n}\, e^{-i\omega t} \right] \sqrt{\frac{\hbar}{2m\omega}}$$

Write $c_i = |c_i| e^{i\phi_i}$ $\forall i \in \mathbb{N}$

$$\Rightarrow \langle x \rangle = \sum_{n\in\mathbb{N}} \sqrt{n}\, |c_n||c_{n-1}| \Big(\underbrace{e^{i(\omega t + \phi_{n-1} - \phi_n)} + e^{-i(\omega t + \phi_{n-1} - \phi_n)}}_{2\cos(\omega t + \phi_{n-1} - \phi_n)} \Big) \sqrt{\frac{\hbar}{2m\omega}} \qquad (*)$$

Obviously $\langle x \rangle (t=0) = \sqrt{\frac{2\hbar}{2m\omega}} \sum_{n\in\mathbb{N}} \sqrt{n}\, |c_n||c_{n-1}| \cos(\phi_{n-1} - \phi_n) \equiv \langle x \rangle_0$

Momentum op.: $p\psi_n(x) = -i\hbar \sqrt{\frac{m\omega}{\hbar}} \frac{d}{d\xi} C_n H_n(\xi) e^{-\xi^2/2} = -i\hbar \sqrt{\frac{m\omega}{\hbar}} C_n \left(2n H_{n-1}(\xi) - \xi H_n(\xi) \right) e^{-\xi^2/2}$

$$= i\hbar \frac{m\omega}{\hbar} x \psi_n(x) - i\hbar \sqrt{\frac{m\omega}{\hbar}} \sqrt{2n}\, \psi_{n-1}(x)$$

$$\Rightarrow \langle \psi_n | p\, \psi_{n'} \rangle = i\hbar \sqrt{\frac{m\omega}{\hbar}} \left[\delta_{n',n-1} \sqrt{\frac{n}{2}} - \delta_{n,n'-1} \sqrt{\frac{n'}{2}} \right]$$

$$\langle p \rangle = \langle \psi(x,t) | p\, \psi(x,t) \rangle = i\hbar \sqrt{\frac{m\omega}{\hbar}} \sum_{n,n'} c_n^* c_{n'} e^{-i(n'-n)\omega t} \left[\delta_{n',n-1} \sqrt{\frac{n}{2}} - \delta_{n,n'-1} \sqrt{\frac{n'}{2}} \right]$$

$$= i\hbar \sqrt{\frac{m\omega}{\hbar}} \sum_{n\in\mathbb{N}} \left[c_n^* c_{n-1} e^{i\omega t} \sqrt{\frac{n}{2}} - c_{n-1}^* c_n e^{-i\omega t} \sqrt{\frac{n}{2}} \right]$$

$$= -\sqrt{2\hbar m\omega} \sum_{n\in\mathbb{N}} \sqrt{n}\, |c_n||c_{n-1}| \sin(\omega t + \phi_{n-1} - \phi_n)$$

$$\Rightarrow \langle p\rangle(0) = -\sqrt{2\hbar m\omega} \sum_{n\in\mathbb{N}} \sqrt{n}\,|c_n||c_{n-1}| \sin(\phi_{n-1}-\phi_n) \equiv \langle p\rangle_0$$

and $\langle p\rangle = m \frac{d}{dt}\langle x\rangle$ from (*)

Also from (*): $\langle x\rangle = \langle x\rangle_0 \cos\omega t + \frac{\langle p\rangle_0}{m\omega} \sin\omega t$

(b) $p^2 = 2m(H-V) = 2mH - m^2\omega^2 x^2$ as operators

$$\Rightarrow \langle \psi_n | p^2 \psi_{n'}\rangle = 2m\langle\psi_n | H\psi_{n'}\rangle - m^2\omega^2\langle\psi_n|x^2\psi_{n'}\rangle$$

$\Rightarrow \langle p^2\rangle = \langle\psi_n|p^2\psi_n\rangle = \underbrace{2m\hbar\omega(n+\tfrac{1}{2})}_{2m\langle H\rangle} - \underbrace{\hbar m\omega\tfrac{1}{2}(2n+1)}_{2m\langle V\rangle} = m\hbar\omega(n+\tfrac{1}{2})$ for diagonal case $n=n'$

In particular $\langle T\rangle = \langle V\rangle$

Virial theorem: $\langle T\rangle = \frac{1}{2}\langle x \frac{dV}{dx}\rangle = \langle V\rangle$ for harm. osc. ✓

For case $n\neq n'$: $\langle\psi_n|p^2\psi_{n'}\rangle = -m^2\omega^2\langle\psi_n|x^2\psi_{n'}\rangle$, cf. [2](c) for HW VI, SPRING 2015

[4] (a) $$W^*(\vec r,\vec p,t) = \left(\int \psi^*(\vec r - \tfrac{\vec r'}{2})\,\psi(\vec r + \tfrac{\vec r'}{2})\, e^{-\frac{i}{\hbar}\vec p\cdot\vec r'}\, d^3r'\right)^* \frac{1}{(2\pi\hbar)^3}$$

$$= \int \psi^*(\vec r + \tfrac{\vec r'}{2})\,\psi(\vec r - \tfrac{\vec r'}{2})\, e^{\frac{i}{\hbar}\vec p\cdot\vec r'}\, d^3r' \frac{1}{(2\pi\hbar)^3}$$

$$\underset{\vec r' = -\tilde{\vec r}}{=} \int \psi^*(\vec r - \tfrac{\tilde{\vec r}}{2})\,\psi(\vec r + \tfrac{\tilde{\vec r}}{2})\, e^{-\frac{i}{\hbar}\vec p\cdot\tilde{\vec r}}\, \underset{(2\pi\hbar)^3}{\underset{\uparrow}{d^3\tilde r}} = W(\vec r,\vec p,t)$$

(b) $$W^2 = \frac{1}{(2\pi\hbar)^6}\left|\int \psi^*(\vec r - \tfrac{\vec r'}{2})\,\psi(\vec r + \tfrac{\vec r'}{2})\, e^{-\frac{i}{\hbar}\vec p\cdot\vec r'}\, d^3r'\right|^2$$

$$\underset{\text{Schwartz}}{\leq} \frac{1}{(2\pi\hbar)^6} \underbrace{\int |\psi(\vec r - \tfrac{\vec r'}{2})|^2 d^3r'}_{2^3} \underbrace{\int |\psi(\vec r + \tfrac{\vec r'}{2})|^2 d^3r'}_{2^3}$$

$$= \left(\frac{2}{h}\right)^6$$

(c) $$\int W(\vec r,\vec p,t)\, W'(\vec r,\vec p,t)\, d^3r\, d^3p$$

$$= \frac{1}{(2\pi\hbar)^6}\int d^3r\, d^3p\, d^3r'\, d^3r''\ \psi^*(\vec r - \tfrac{\vec r'}{2})\,\psi(\vec r + \tfrac{\vec r'}{2})\,\psi'^*(\vec r - \tfrac{\vec r''}{2})\,\psi'(\vec r + \tfrac{\vec r''}{2})\ e^{-\frac{i}{\hbar}\vec p\cdot(\vec r' + \vec r'')}$$

$$= \frac{1}{(2\pi\hbar)^3}\int d^3r\, d^3r'\ \psi^*(\vec r - \tfrac{\vec r'}{2})\,\psi(\vec r + \tfrac{\vec r'}{2})\,\psi'^*(\vec r + \tfrac{\vec r'}{2})\,\psi'(\vec r - \tfrac{\vec r'}{2})$$

$$\underset{\substack{\hat r = \vec r - \frac{\vec r'}{2} \\ \hat{\hat r} = \vec r + \frac{\vec r'}{2}}}{=} \frac{1}{(2\pi\hbar)^3}\int d^3\hat r\ \psi^*(\hat r)\,\psi'(\hat r) \int d^3\hat{\hat r}\ \psi'^*(\hat{\hat r})\,\psi(\hat{\hat r}) = \frac{1}{(2\pi\hbar)^3}\left|\langle\psi'|\psi\rangle\right|^2$$