PHYS 606 – Spring 2015 – Homework III – Solution

Problem [1]

(a) Induction: eqs (2) + (3) obviously true for $n=1$; suppose they are true for $n-1$

Then
$$\left\{\begin{aligned} [F,G^n] &= G^{n-1}[F,G] + [F,G^{n-1}]G = G^{n-1}[F,G] + (n-2)G^{n-2}[F,G]G \\ [F^n,G] &\underset{\substack{\uparrow \\ \text{Product rule}}}{=} F^{n-1}[F,G] + [F^{n-1},G]F = F^{n-1}[F,G] + (n-2)F^{n-2}[F,G]F \end{aligned}\right\}$$

G commutes w/ $[F,G]$

$$= \begin{cases} (n-1)G^{n-1}[F,G] \\ (n-1)F^{n-1}[F,G] \end{cases}$$

Direct calculation is also possible.

(b)
$$[F,[G,H]] = [F,GH] - [F,HG] \underset{\substack{| \\ \text{Product rule}}}{=} G[F,H] + [F,G]H - H[F,G] - [F,H]G$$
$$= [G,[F,H]] - [H,[F,G]] = -[H,[F,G]] - [G,[H,F]]$$

Problem [2]

$$\left[F_{\vec r}(\vec r),\, G_{\vec r}(\vec p)\right] f(\vec r) = (2\pi\hbar)^{-3/2} \int_{\mathbb{R}^3} \left[F(\vec r),\, G(-i\hbar\nabla_r)\right] \hat f(\vec p)\, e^{\frac{i}{\hbar}\vec r\cdot\vec p}\, d^3p$$

$$= (2\pi\hbar)^{-3/2} \int_{\mathbb{R}^3} \Big(F(\vec r)\, \hat f(\vec p)\, \underbrace{G(\vec p)\, e^{\frac{i}{\hbar}\vec r\cdot\vec p}}_{\substack{\text{power series of}\\ G \text{ has acted on}\\ \text{phase factor}}} - G(-i\hbar\nabla_r)\, \hat f(\vec p)\, \underbrace{F(-i\hbar\nabla_p)\, e^{\frac{i}{\hbar}\vec r\cdot\vec p}}_{\substack{\text{power series}\\ \text{of } F \text{ acting}\\ \text{on phase factor}}} \Big)\, d^3p$$

$$= (2\pi\hbar)^{-3/2} \int_{\mathbb{R}^3} \Big(G(\vec p)\, \hat f(\vec p)\, \underline{F(-i\hbar\nabla_p)}\, e^{\frac{i}{\hbar}\vec r\cdot\vec p} - \hat f(\vec p)\, F(-i\hbar\nabla_p)\, \underline{G(\vec p)}\, e^{\frac{i}{\hbar}\vec r\cdot\vec p} \Big)\, d^3p$$

$$= (2\pi\hbar)^{-3/2} \int_{\mathbb{R}^3} \Big(\underbrace{\big(F(+i\hbar\nabla_p)\, G(\vec p)\, \hat f(\vec p)\big)}_{\substack{\text{multiple partial integrations}\\ \text{according to the power series}\\ \text{of } F\text{, boundary terms vanish}\\ \text{for sufficiently fast falling}\\ \text{functions.}}} e^{\frac{i}{\hbar}\vec r\cdot\vec p} - G(\vec p)\, \underbrace{\big(F(+i\hbar\nabla_p)\, \hat f(\vec p)\big)}_{\substack{\text{partial}\\ \text{integrations}}} e^{\frac{i}{\hbar}\vec r\cdot\vec p} \Big)\, d^3p$$

$$= (2\pi\hbar)^{-3/2} \int_{\mathbb{R}^3} \left[F_{\vec p}(\vec r),\, G_{\vec p}(\vec p)\right] \hat f(\vec p)\, e^{\frac{i}{\hbar}\vec r\cdot\vec p}\, d^3p$$

[3] $\frac{d}{dt}\langle p\rangle \underset{\text{(Ehrenfest)}}{=} \langle F\rangle = 0$ (free particle) $\Rightarrow \langle p\rangle(t) = \underbrace{\langle p\rangle(0)}_{=:\,p_0 \text{ short notation}} = \text{const.}$

$$\frac{d}{dt}\langle x\rangle = \frac{\langle p\rangle}{m} = \frac{p_0}{m}$$

$$\Rightarrow \langle x\rangle(t) = \frac{p_0}{m}t + x_0 \qquad x_0 = \langle x\rangle(0)$$

$$\frac{d}{dt}\langle p^2\rangle = \frac{1}{i\hbar}\langle[p, \underbrace{T}_{\frac{p^2}{2m}}]\rangle = 0 \qquad \Rightarrow \langle p^2\rangle(t) = \langle p^2\rangle(0) = \text{const.}$$

$$\Rightarrow (\Delta p)^2(t) = \langle(p-\langle p\rangle)^2\rangle = \langle p^2\rangle(t) - (\langle p\rangle(t))^2 = \langle p^2\rangle(0) - p_0^2 = (\Delta p)^2(0) = \text{const.}$$

$$\frac{d}{dt}\langle x^2\rangle = \frac{1}{i\hbar}\langle[x^2, \tfrac{p^2}{2m}]\rangle = \frac{1}{i\hbar}\langle p[x^2, \tfrac{p}{2m}] + [x^2, \tfrac{p}{2m}]p\rangle$$

$$= \frac{1}{i\hbar}\langle px[x,p] + p[x,p]x + x[x,p]p + [x,p]xp\rangle \frac{1}{2m}$$

$$= \frac{1}{m}\langle px + xp\rangle$$

$$\frac{d}{dt}\langle px\rangle = \frac{1}{i\hbar}\langle[px, T]\rangle \overset{[2](a)}{=} 2T = \text{const.} = \frac{d}{dt}\langle xp\rangle$$

$$\Rightarrow \langle px + xp\rangle(t) = \frac{2\langle p^2\rangle(0)}{m}t + \langle px + xp\rangle(0)$$

$$\Rightarrow \langle x^2\rangle = \frac{\langle p^2\rangle(0)}{m^2}t^2 + \frac{1}{m}\langle px + xp\rangle(0)\,t + \langle x^2\rangle(0)$$

$$\Rightarrow (\Delta x)^2(t) = \langle x^2\rangle(t) - (\langle x\rangle(t))^2 = \frac{\langle p^2\rangle(0) - p_0^2}{m^2}t^2 + \frac{t}{m}\left(\langle px + xp\rangle(0) - 2x_0p_0\right) + \langle x^2\rangle(0) - x_0^2$$

$$= \frac{(\Delta p)^2(t)}{m^2}t^2 + \frac{2}{m}\left(\frac{1}{2}\langle px + xp\rangle(0) - \langle p\rangle(0)\langle x\rangle(0)\right)t + (\Delta x)^2(0)$$

[4] (a) Gauss: $(\Delta x)^2(0) = \sigma^2$, $(\Delta p)^2(0) = \frac{\hbar^2}{4\sigma^2}$ (HW I, [3])

$$\langle x\rangle(0) = x_0 \quad , \quad \langle p\rangle(0) = \hbar k_0 \equiv p_0$$

$$\langle xp\rangle(0) = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{\mathbb{R}} e^{-\frac{(x-x_0)^2}{4\sigma^2}} e^{-\frac{i}{\hbar}p_0x}\, x\left(-i\hbar\frac{d}{dx}\right) e^{-\frac{(x-x_0)^2}{4\sigma^2}} e^{+\frac{i}{\hbar}p_0x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma}\int_{\mathbb{R}} e^{-\frac{(x-x_0)^2}{2\sigma^2}}\, x\left(p_0 + i\hbar\frac{x-x_0}{2\sigma^2}\right) dx$$

$$\underset{u=x-x_0}{=} \frac{1}{\sqrt{2\pi}\,\sigma} \int e^{-\frac{u^2}{2\sigma^2}} \left(\cancel{u p_0} + i\hbar \frac{u^2}{2\sigma^2} + x_0 p_0 + i\hbar x_0 \cancel{\frac{u}{2\sigma^2}} \right) d$$

$$= x_0 p_0 + i\hbar \frac{\sigma^2}{2\sigma^2} = x_0 p_0 + \frac{1}{2} i\hbar$$

$\langle px \rangle(0) =$ same as above + term with $\left(-i\hbar \frac{d}{dx}\right)$ acting on x

$$= x_0 p_0 + \frac{1}{2} i\hbar + \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} e^{-\frac{u^2}{2\sigma^2}} (-i\hbar) = x_0 p_0 - \frac{1}{2} i\hbar$$

$$\Rightarrow \frac{1}{2} \langle px + xp \rangle (0) = x_0 p_0$$

$$\Rightarrow (\Delta x)^2(t) = \frac{\hbar^2}{4\sigma^2 m^2} t^2 + \sigma^2 \quad \text{and } (\Delta p)^2(t) = \frac{\hbar^2}{4\sigma^2}$$

(b) $$(\Delta p)^2(t) \overset{HW\,II,\,[1](a)}{=} \frac{1}{\sqrt{2\pi}\,\hat{\sigma}} \int e^{-\frac{(p-p_0)^2}{4\hat{\sigma}^2}} \left| e^{-\frac{i}{\hbar}(p-p_0)x_0} e^{-i\omega(p)t} \right|^2 (p-p_0)^2 \, dp \quad \left(\hat{\sigma} = \frac{\hbar}{2\sigma} \right)$$

$$= (\Delta p)^2(0) = \hat{\sigma}^2 = \frac{\hbar^2}{4\sigma^2}$$

HW II, [1](a): $$|\psi(x,t)|^2 = \frac{1}{\sqrt{2\pi}} \frac{\sigma}{\sqrt{\sigma^4 + \frac{\hbar^2 t^2}{4m^2}}} e^{-\frac{(x-x_0-v_{gr}t)^2 \sigma^2}{2\left(\sigma^4 + \frac{\hbar^2 t^2}{4m^2}\right)}} \qquad v_{gr} = \frac{p_0}{m}$$

$$\langle x \rangle (t) \underset{x-x_0-v_{gr}t=:u}{=} \frac{1}{\sqrt{2\pi}} \frac{\sigma}{\sqrt{\sigma^4 + \frac{\hbar^2 t^2}{4m^2}}} \int_{\mathbb{R}} du \, \underbrace{(x_0 + v_{gr} t + \cancel{u})}_{=x} e^{-\frac{u^2 \sigma^2}{2\left(\sigma^4 + \frac{\hbar^2 t^2}{4m^2}\right)}} = x_0 + v_{gr} t$$

$$\langle x^2 \rangle (t) = \frac{1}{\sqrt{2\pi}} \frac{\sigma}{\sqrt{\sigma^4 + \frac{\hbar^2 t^2}{4m^2}}} \int_{\mathbb{R}} du \left((x_0 + v_{gr} t)^2 + \cancel{2u(x_0 + v_{gr} t)} + u^2 \right) e^{-\frac{u^2 \sigma^2}{2\left(\sigma^4 + \frac{\hbar^2 t^2}{4m^2}\right)}}$$

$$= (x_0 + v_{gr} t)^2 + \frac{\sigma^4 + \frac{\hbar^2 t^2}{4m^2}}{\sigma^2}$$

$$\Rightarrow \langle (x - \langle x \rangle)^2 \rangle = \langle x^2 \rangle (t) - (\langle x \rangle (t))^2 = \sigma^2 + \frac{\hbar^2}{4\sigma^2 m^2} t^2$$