PHYS 606 – Spring 2015 – Homework I

Problem [1] (equation numbers refer to my online manuscript)

<u>Proof</u>: We have

$$\hat{\delta}_y(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x-y) e^{-ikx} = \frac{1}{\sqrt{2\pi}} e^{-iky} \tag{1.37}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \hat{\delta}_y(k) dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-y)} dk = \delta(x-y) \tag{1.38}$$

For the last $=$ sign one needs to check that the integral expression on its left hand side satisfies the defining properties of δ, i.e. it needs to be integrated over test functions to recover (1.33),

(1.34). The proof goes as follows. We will only check property (1.33), as Eq. (1.34) can be quite readily seen. It is then also sufficient to show it for $I = \mathbb{R}$. Then using the antisymmetry of the sin-function twice the k-integral is

$$\begin{aligned} \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ik(x-y)} f(x)\, dk dx &= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{x-y} \sin k(x-y) \Big|_{k=-\infty}^{k=\infty} f(x)\, dx \\ &= \frac{k}{\pi} \int_{\mathbb{R}} \lim_{k\to\infty} \frac{\sin k(x-y)}{k(x-y)} f(x)\, dx\,. \end{aligned} \tag{1.39}$$

We realize that the sinc-function in the limit $k \to \infty$ becomes more and more narrow so that all its strength will lie at $x = y$. We can thus expand f around $u = 0$ and just replace it by $f(y)$. The integral then becomes independent of k and we easily substitute a new integration variable $u \equiv k(x-y)$ and using the well-known normalization of the sinc-function we get

$$\frac{k}{\pi} \int_{\mathbb{R}} \lim_{k\to\infty} \operatorname{sinc} k(x-y)\, f(y) dx = f(y) \frac{1}{\pi} \int_{\mathbb{R}} \operatorname{sinc} u\, du = f(y)\,. \tag{1.40}$$

I.e. the reverse Fourier transform of the plane wave has indeed the properties of the δ-function.

Problem [2]

(a) Hamilton fct. $H = \frac{p^2}{2m} + \frac{k}{2}x^2 = E \Rightarrow \frac{p^2}{2mE} + \frac{x^2}{\frac{2E}{k}} = 1$

$\Rightarrow$ Phase space motion is ellipse with semi axes $\sqrt{2mE}$ and $\sqrt{\frac{2E}{k}}$

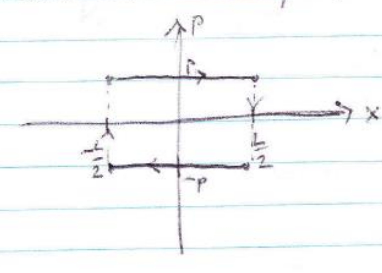
Bohr – Sommerfeld: $\underbrace{\oint p\,dq}_{\text{area of ellipse}} = \pi\sqrt{2mE}\sqrt{\frac{2E}{k}} = 2\pi\frac{E}{\omega}$ with $\omega = \sqrt{\frac{k}{m}}$

On the other hand $\oint p\,dq \doteq nh$

$\Rightarrow E = n\,\frac{h\omega}{2\pi} = n\hbar\omega$

All these energy levels are off by $\frac{1}{2}\hbar\omega$ from the full QM result, but the radiation spectrum (involving ΔE) can be predicted accurately.

(b) Consider particle with momentum p (moving right); reflected at $x = +\frac{L}{2}$ to obtain momentum $-p$ (energy conserved); another reflection $-p \to p$ at $x = -\frac{L}{2}$



$\oint p\,dq = 2pL \doteq nh$

$\Rightarrow E = \frac{p^2}{2m} = n^2\,\frac{h^2}{8mL^2}$

Same as the full QM result!

[3] (a) $$|C|^2 \int_{\mathbb{R}} e^{-\frac{(x-x_0)^2}{2\sigma^2}} dx \underset{z=\frac{x-x_0}{\sqrt{2}\sigma}}{=} |C|^2 \int_{\mathbb{R}} e^{-z^2} dz \cdot \sqrt{2}\sigma = |C|^2 \sqrt{2\pi}\,\sigma$$

One bonus point if you calculated that integral yourself:

$$\int_{\mathbb{R}} e^{-z^2} dz = \left(\int_{\mathbb{R}^2} e^{-r^2} d^2r \right)^{1/2} = \left(2\pi \int_0^\infty r e^{-r^2} dr \right)^{1/2} \underset{u=r^2}{=} \left(\pi \int_0^\infty e^{-u} du \right)^{1/2} = \sqrt{\pi}$$

$$\Rightarrow C = \frac{1}{\sqrt{\sqrt{2\pi}\,\sigma}} \qquad (\times \text{ a phase which we ignore here})$$

(b) $$\langle x \rangle = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{\mathbb{R}} x\, e^{-\frac{(x-x_0)^2}{2\sigma^2}} dx \underset{u=\frac{x-x_0}{\sqrt{2}\sigma}}{=} \frac{1}{\sqrt{2\pi}\,\sigma} \int_{\mathbb{R}} (\sqrt{2}\sigma u + x_0) e^{-u^2} \sqrt{2}\sigma\, du$$

$$= \frac{x_0}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-u^2} du = x_0$$

$\int u e^{-u^2} du = 0$

$$(\Delta x)^2 = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{\mathbb{R}} (x-x_0)^2 e^{-\frac{(x-x_0)^2}{2\sigma^2}} dx \underset{u=\frac{x-x_0}{\sqrt{2}\sigma}}{=} \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} 2\sigma^2 u^2 e^{-u^2} du = -\frac{\sigma^2}{\sqrt{\pi}} \int_{\mathbb{R}} u \frac{d}{du} e^{-u^2} du$$

$$\underset{\text{part. int}}{=} \underbrace{\left[-\frac{\sigma^2}{\sqrt{\pi}} u e^{-u^2} \right]_{-\infty}^{+\infty}}_{0} + \frac{\sigma^2}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-u^2} du = \sigma^2$$

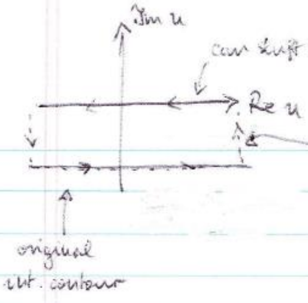
Center x_0 and width σ are equal to "average x" and $\sqrt{\text{variance}} = \Delta x$, resp.

(c) $$\hat{f}(k) = (2\pi)^{-1/2} \frac{1}{\sqrt{\sqrt{2\pi}\,\sigma}} \int_{\mathbb{R}} e^{-\frac{(x-x_0)^2}{4\sigma^2}} e^{-ikx} dx$$

$$= \frac{1}{(2\pi)^{3/4} \sigma^{1/2}} e^{-ikx_0} \int_{\mathbb{R}} e^{-\frac{x^2}{4\sigma^2}} e^{-ikx} dx$$

Complete square → $$= \frac{1}{(2\pi)^{3/4} \sigma^{1/2}} e^{-ikx_0} \int_{\mathbb{R}} e^{-\left(\frac{x}{2\sigma} + ik\sigma\right)^2} dx \; e^{-\sigma^2 k^2}$$

$$\underset{u=\frac{x}{2\sigma}+ik\sigma}{=} \frac{2^{1/4} \sigma^{1/2}}{\pi^{3/4}} e^{-ikx_0} e^{-\sigma^2 k^2} \int_{-\infty-2ik\sigma^2}^{+\infty-2ik\sigma^2} e^{-u^2} du = \sqrt[4]{\frac{2}{\pi}} \sqrt{\sigma}\, e^{-ikx_0} e^{-\sigma^2 k^2}$$



$\oint e^{-u^2} du = 0$ for this closed contour since e^{-u^2} analytic everywhere

so $$\int_{-\infty - 2ik\sigma^2}^{+\infty - 2ik\sigma^2} e^{-u^2} du = \int_{-\infty}^{+\infty} e^{-u^2} du = \sqrt{\pi}$$

We can write this in "standard form" with a width in momentum space

$\hat{\sigma} = \frac{1}{2\sigma}$; then $\quad \hat{f}(k) = \frac{1}{\sqrt{\sqrt{2\pi}\,\hat{\sigma}}}\, e^{-ikx_0}\, e^{-\frac{k^2}{4\hat{\sigma}^2}}$

Summary:	Gauss centered around 0, width σ	$\xleftrightarrow{FT}$	Gauss centered around 0, width $\frac{1}{2\sigma}$
	Displacement by x_0	$\xleftrightarrow{FT}$	phase factor e^{-ikx_0}

Using the result from (b): $\quad \langle k \rangle = 0$

(phase e^{-ikx_0} drops out) $\quad (\Delta k)^2 = \langle k^2 \rangle = \sigma_k^2 = \frac{1}{4\sigma^2}$

$\Rightarrow \Delta x\, \Delta k = \sigma \sigma_k = \frac{1}{2}$

Problem [4]

Hamilton fct. $H(x,p) = \frac{p^2}{2m} - bx$

Hamilton - Jacobi: $\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 - bx + \frac{\partial S}{\partial t} = 0$ with $p = \frac{\partial S}{\partial x}$

Since $\frac{\partial H}{\partial t} = 0$ time is separable: $S = W(x) - Et$

$\Rightarrow \frac{1}{2m}\left(\frac{dW}{dx}\right)^2 = E + bx \Rightarrow \frac{dW}{dx} = \pm\sqrt{2m(E+bx)}$

$\Rightarrow W(x) = \pm \frac{1}{3mb}\left[2m(E+bx)\right]^{3/2} + \text{const.}$

$\Rightarrow S(x,t) = \pm \frac{1}{3mb}\left[2m(E+bx)\right]^{3/2} - Et + \text{const.}$

E is constant of motion choose it a the variable after canonical transf.

$\Rightarrow$ Associated momentum $\beta = \frac{\partial S}{\partial \alpha} = \text{const.}$ and $\beta = \pm\frac{1}{b}\sqrt{2m(E+bx)} - t$

$\Rightarrow x = \frac{b}{2m}(t+\beta)^2 - \frac{E}{b}$

Initial conditions: $x(0) = \frac{b\beta^2}{2m} - \frac{E}{b} \overset{!}{=} x_0$ $\quad\dot{x}(0) = \frac{b\beta}{m} \overset{!}{=} v_0$

$\Rightarrow \beta = \frac{m}{b}v_0$ and $E = \frac{b^2}{2m}\frac{m^2}{b^2}v_0^2 - bx_0 = \frac{1}{2}mv_0^2 - bx_0$

$\Rightarrow x(t) = \frac{b}{2m}\left(t + \frac{m}{b}v_0\right)^2 - \frac{mv_0^2}{2b} + x_0$