PHYS 606 — Spring 2015 — Midterm Exam — Solution

[1] (a) In coord. space $\vec{L} = \vec{r} \times (-i\hbar \nabla_r)$; $T = -\frac{\hbar^2}{2m}\Delta_r$

$$[L_x, T] = (-i\hbar)\left(-\frac{\hbar^2}{2m}\right)\left[y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]$$

$$= (-i\hbar)\left(-\frac{\hbar^2}{2m}\right)\Big(y\frac{\partial}{\partial z}\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x^2} y \frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2} z \frac{\partial}{\partial y}$$

$$+ y\frac{\partial}{\partial z}\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial y^2} y\frac{\partial}{\partial z} - z\frac{\partial^3}{\partial y^3} + \frac{\partial^2}{\partial y^2} z \frac{\partial}{\partial y}$$

$$+ y\frac{\partial^3}{\partial z^3} - \frac{\partial^2}{\partial z^2} y \frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial z^2} z \frac{\partial}{\partial y}\Big)$$

$$= (-i\hbar)\left(-\frac{\hbar^2}{2m}\right)\left(-\frac{\partial}{\partial y}\frac{\partial}{\partial z} + \frac{\partial}{\partial z}\frac{\partial}{\partial y}\right) = 0$$

Similar for $[L_y, T] = 0 = [L_z, T]$

<u>OR</u> break it down to fundamental commutators $[r_i, p_j] = i\hbar\delta_{ij}$ etc.

(b) Commutator has to be the same in $\vec{r}$- and $\vec{p}$-space if just a number.

<u>OR</u> calculate explicitly as above <u>OR</u> break it down to fundamental known commutators like $[r_i, p_j]$

(c) $$\frac{d}{dt}\langle \vec{L} \rangle = \frac{1}{i\hbar}\langle [\vec{L}, H] \rangle + \underbrace{\left\langle \frac{\partial \vec{L}}{\partial t} \right\rangle}_{=0} \underset{(a)}{=} \frac{1}{i\hbar}\langle [\vec{L}, V] \rangle$$

$$= -\langle [\vec{r} \times \nabla, V] \rangle = -\langle \vec{r} \times (\nabla V) \rangle = \langle \vec{r} \times \vec{F} \rangle$$

②

[2] (a) $\psi = A e^{\frac{i}{\hbar} S}$ in $i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V\psi$

$$\Rightarrow \quad i\hbar \frac{\partial A}{\partial t}\frac{1}{A}\psi \cancel{+} - \frac{\partial S}{\partial t}\psi = -\frac{\hbar^2}{2m}(\Delta A)\frac{1}{A}\psi - \frac{i\hbar}{m}(\nabla A)\cdot(\nabla S)\frac{1}{A}\psi$$

$$- \frac{i\hbar}{2m}\Delta S\,\psi + \frac{1}{2m}(\nabla S)^2\psi + V'\psi - iV''\psi$$

Separate imaginary + real part:

$$(R) \qquad -\frac{\partial S}{\partial t} = -\frac{\hbar^2}{2m}\frac{1}{A}\Delta A + \frac{1}{2m}(\nabla S)^2 + V'$$

$$(I) \qquad \frac{\hbar}{A}\frac{\partial A}{\partial t} = -\frac{\hbar}{m}\frac{1}{A}(\nabla A)\cdot(\nabla S) - \frac{\hbar}{2m}\Delta S - V''$$

$$(R) \underset{\hbar \to 0}{\Longrightarrow} \quad \frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 + V' = 0$$

the usual Hamilton-Jacobi equation with $\nabla S \to \vec{p}$

$$(I) \underset{\rho = A^2}{\Longrightarrow} \quad \frac{\partial \rho}{\partial t} = -\frac{1}{m}(\nabla S)\cdot(\nabla\rho) - \frac{1}{m}(\Delta S)\rho - \frac{2\rho}{\hbar}V''$$

$$\underset{\frac{1}{m}\nabla S \to \frac{\vec{p}}{m} = \vec{v}}{\Longrightarrow} \quad \underbrace{\frac{\partial \rho}{\partial t} + (\nabla\vec{v})\rho + \vec{v}\cdot\nabla\rho}_{\text{the usual terms in the cont. equation}} + \underbrace{\frac{2}{\hbar}V''\rho}_{\text{additional term}} = 0$$

(b) The additional term is a source term that can decrease or increase particle number.

[3] (a) outside the well $\psi_n = 0$ because otherwise infinite energy

boundary condition: $\psi_n = 0$ @ $x = 0$, $x = L$

inside the well: $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_n = E_n \psi_n$

$\Rightarrow \quad \psi_n = C_n \sin\frac{n\pi}{L}x$ using the boundary conditions $n = 1, 2, 3, \ldots$

$$E_n = n^2 \frac{\hbar^2 \pi^2}{2mL^2}$$ non-degenerate eigenvalues of H

Normalization: $\int_0^L |C_n|^2 \sin^2 \frac{n\pi}{L} x \, dx = |C_n|^2 \underbrace{\int_0^{n\pi} \sin^2 \phi \, \frac{L}{n\pi} \, d\phi}_{\frac{1}{2} n\pi} = |C_n|^2 \frac{L}{2}$

$$\Rightarrow \psi_n = \sqrt{\frac{2}{L}} \sin \frac{n\pi}{L} x$$

(b) $\psi(x,0) = \psi_1(x)$ ← this is called ψ_0 in the problem sheet. in any case this is the ground state $\sqrt{\frac{2}{L}} \sin \frac{\pi x}{L}$

Ansatz $\psi(x,t) = N(t)\psi_1(x)$ into $i\hbar \frac{\partial \psi}{\partial t} = H\psi$

Outside the well still $\psi(x,t) = 0$ because energy needs to be finite.

Inside: $i\hbar \frac{\partial N}{\partial t} \psi_1(x) = \underbrace{- \frac{\hbar^2}{2m} (\Delta \psi_1)}_{= E_0 \psi_1 \text{ from (a)}} N(t) + i V_s \psi_1 N(t)$

$$\Rightarrow \quad \frac{\partial N}{\partial t} = \left(-\frac{i}{\hbar} E_0 + \frac{1}{\hbar} V_s\right) N \quad \Rightarrow N(t) = C' e^{-\frac{i}{\hbar}E_0 t + \frac{1}{\hbar} V_s t}$$

From initial condition $N(0) = 1 \Rightarrow C' = 1$

$$\Rightarrow \psi(x,t) = \psi_1(x)\, e^{-\frac{i}{\hbar}E_0 t} \, e^{\frac{1}{\hbar} V_s t}$$

↑ usual time evolution for stationary state

↖ normalization increases/decreases due to source term

[4] (a) $\langle \psi | \psi \rangle = |C|^2 \big(\underbrace{\langle \psi_1|\psi_1\rangle}_{=1} + \underbrace{\langle \psi_1|\psi_2\rangle}_{=0} + \underbrace{\langle \psi_2|\psi_1\rangle}_{=0} + \underbrace{\langle \psi_2|\psi_2\rangle}_{=1} \big) = 2|C|^2$

$$\Rightarrow C = \frac{1}{\sqrt{2}}$$

$$\psi(x,t) = \frac{1}{\sqrt{2}} \left(\psi_1(x)\, e^{-\frac{i}{\hbar}E_1 t} + \psi_2(x)\, e^{-\frac{i}{\hbar}E_2 t} \right)$$

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(b) $\langle E\rangle = \langle\psi|H\psi\rangle \underset{SE}{=} \langle\psi|i\hbar\frac{\partial\psi}{\partial t}\rangle = \frac{1}{2}\int\left(\psi_1^* e^{\frac{i}{\hbar}E_1 t} + \psi_2^* e^{\frac{i}{\hbar}E_2 t}\right)\left(E_1\psi_1 e^{-\frac{i}{\hbar}E_1 t} + E_2\psi_2 e^{-\frac{i}{\hbar}E_2 t}\right)dx$

$= \frac{1}{2}\left(E_1\langle\psi_1|\psi_1\rangle + E_2\langle\psi_2|\psi_2\rangle\right) = \frac{1}{2}(E_1+E_2)$

mixed terms vanish

(c) $\Delta E = \langle(H-\langle E\rangle)^2\rangle^{\frac{1}{2}} = \left(\langle E^2\rangle - \langle E\rangle^2\right)^{\frac{1}{2}}$

$= \left(\frac{1}{2}\int\left(\psi_1^* e^{\frac{i}{\hbar}E_1 t} + \psi_2^* e^{\frac{i}{\hbar}E_2 t}\right)\left(E_1^2\psi_1 e^{-\frac{i}{\hbar}E_1 t} + E_2\psi_2 e^{-\frac{i}{\hbar}E_2 t}\right)dx - \langle E\rangle^2\right)^{1/2}$

$= \left(\frac{1}{2}\left(E_1^2 + E_2^2\right) - \frac{1}{4}(E_1+E_2)^2\right)^{\frac{1}{2}} = \sqrt{\frac{1}{4}(E_1-E_2)^2} = \frac{1}{2}(E_2-E_1)$

[5] $T\psi + V\psi = E\psi \Rightarrow \langle\psi|T\psi\rangle + \cancel{V\langle\psi|\psi\rangle} = E\underbrace{\langle\psi|\psi\rangle}_{=1}$

if normalized to one (argument also goes through if not norma :)

$\Rightarrow E = \langle\psi|V\psi\rangle + \langle\psi|T\psi\rangle$

$\langle\psi|V\psi\rangle = \int|\psi|^2 V(x)\,dx \geq \int|\psi|^2 V_0\,dx = V_0$

$\langle\psi|T\psi\rangle = \langle\phi|T\phi\rangle = \int|\phi|^2\frac{p^2}{2m}dp \geq 0$

mom-space representation

$\Rightarrow E \geq V_0 + 0 = V_0$