I.14 The Wigner Formalism

* Let $\psi(\vec{r},t)$ be the wave fct. of a system (say at a given t)
and $\phi(\vec{p},t)$ be the corresponding wave fct. in momentum space.

The function
$$\boxed{\begin{aligned} W(\vec{r},\vec{p},t) &= \frac{1}{(2\pi\hbar)^3}\int \psi^*(\vec{r}-\tfrac{\vec{r}\,'}{2},t)\,\psi(\vec{r}+\tfrac{\vec{r}\,'}{2},t)\, e^{-\frac{i}{\hbar}\vec{p}\cdot\vec{r}\,'}\, d^3r' \\ &= \frac{1}{(2\pi\hbar)^3}\int \phi^*(\vec{p}-\vec{p}\,')\,\phi(\vec{p}+\vec{p}\,')\, e^{\frac{i}{\hbar}2\vec{r}\cdot\vec{p}\,'}\, d^3p' \end{aligned}}$$

is called the "Wigner distribution" of the state described be ψ (or ϕ).*

* if above integrals exist.

(check that the definition via ϕ and the definition via ψ are equivalent!)

* The Wigner fct. is a phase space distribution that has the full information about the quantum system. ψ (or ϕ) can be recovered through the inverse Wigner (Weyl) transformation

$$\psi^*(\vec{r}\,')\,\psi(\vec{r}) = \int W\left(\frac{\vec{r}+\vec{r}\,'}{2},\vec{p}\right) e^{\frac{i}{\hbar}\vec{p}\cdot(\vec{r}\,'-\vec{r})}\, d^3p$$

The functional form of ψ can then be recovered (e.g by integrating over $\vec{r}\,'$).

* Fundamental properties:

- $W^*(\vec{r},\vec{p},t) = W(\vec{r},\vec{p},t)$ — the Wigner function is real. [HW]
- $\int W(\vec{r},\vec{p},t)\, d^3r\, d^3p = 1$ — Wigner fct. normalized to unity (for wave fcts. normalized to unity) (check!)
- $-\left(\frac{2}{h}\right)^3 \le W(\vec{r},\vec{p},t) \le \left(\frac{2}{h}\right)^3$ — universal bounds [HW]

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* Because of the first and second properties it is tempting to assign the Wigner function an interpretation as a probability density in phase space. However W is not always positive definite, i.e. there can be regions in phase space where $W(\vec{r},\vec{p}) < 0$.
Examples maybe later.

* Recall that classical phase space distributions are "sharp ridges" of the type $f(\vec{r},\vec{p},t) = \delta^{(3)}(\vec{r}-\vec{r}(t))\,\delta^{(3)}(\vec{p}-\vec{p}(t))$
These are precluded by the bound $\left(\frac{2}{h}\right)^3$ for W, but would be allowed in the classical limit $h \to 0$. The Wigner fct thus is "smoother" than a classical distribution and must incorporate the uncertainty principle.

* We can express standard probabilities and probability densities in QM through the Wigner fct.:

- $|\psi(\vec{r})| = \int d^3p\; W(\vec{r},p)$ — $|\phi(\vec{p})|^2 = \int d^3r\; W(\vec{r},\vec{p})$

- $\boxed{\langle A \rangle = \int A(\vec{r},\vec{p})\, W(\vec{r},\vec{p})\, d^3r\, d^3p}$ for any observable A with separable pairs of conjugate variables.

(Check!)

- For two states ψ_1, ψ_2 with Wigner fcts. W_1 and W_2 the probability of overlap is

$$|\langle \psi_2|\psi_1\rangle|^2 = (2\pi\hbar)^3 \int d^3r\, d^3p\;\; W_1(\vec{r},\vec{p})\, W_2(\vec{r},\vec{p})$$

Why? HW

* Free particles lead to a purely classical equation of motion for Wigner fcts:

$$\boxed{\frac{\partial W(\vec{r},\vec{p},t)}{\partial t} + \frac{1}{m}\vec{p}\cdot\nabla_{\vec{r}}\, W(\vec{r},\vec{p},t) = 0}$$

In the general case the equation of motion is more complicated but in the classical limit it can be written as

$$\left[\frac{\partial}{\partial t} + \frac{1}{m}\vec{p}\cdot\nabla_{\vec{r}} - (\nabla_{\vec{r}}V)\cdot\nabla_{\vec{p}}\right] W(\vec{r},\vec{p},t) = \mathcal{O}(\hbar)$$

* Simple example: free Gaussian wave packet (1-D):

$$\psi(x) = \frac{1}{\sqrt{\sqrt{2\pi}\,\sigma}}\, e^{-\frac{(x-x_0)^2}{4\sigma^2}}\, e^{ip_0x}$$

$$\Rightarrow W(x,p) = \frac{1}{2\pi}\frac{1}{\sqrt{2\pi}\,\sigma}\int dy\, e^{-\frac{(x+\frac{y}{2}-x_0)^2}{4\sigma^2}}\, e^{-\frac{(x-\frac{y}{2}-x_0)^2}{4\sigma^2}}\, e^{\frac{i}{\hbar}p_0(x+\frac{y}{2})}\, e^{-\frac{i}{\hbar}p_0(x-\frac{y}{2})}\; e^{-\frac{i}{\hbar}py}$$

$$= \frac{1}{(2\pi)^{3/2}\sigma}\int dy\, e^{-\frac{(x-x_0)^2}{2\sigma^2}}\, e^{-\frac{y^2}{8\sigma^2}}\, e^{-\frac{i}{\hbar}y(p-p_0)}$$

$$= \frac{1}{(2\pi)^{3/2}\sigma}\int dy\, e^{-\frac{(x-x_0)^2}{2}}\, e^{-\frac{1}{2\sigma^2}\left(\frac{y}{2}+i(p-p_0)2\sigma^2\right)^2}\, e^{-2\sigma^2(p-p_0)^2}$$

$$= \frac{1}{\pi}\, e^{-\frac{(x-x_0)^2}{2\sigma^2}}\, e^{-2\sigma^2(p-p_0)^2}$$

In this case the Wigner fct. just resembles a product of $|\psi(\vec{r})|^2$ and $|\phi(\vec{p},t)|^2$. and is positive definite.

Examples with negative WF: HW

* For free particles there is on average one eigenstate per phase space volume h^3.

Why? HW V [3]