Physics 606 (Quantum Mechanics I) — Spring 2015

Midterm Exam

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[1] **The Angular Momentum Operator** (20 points)

Consider a particle of mass m subject to a potential energy $V(\vec{r})$.

(a) (6) Calculate the commutator $[\vec{L}, T]$ of the angular momentum operator $\vec{L} = \vec{r} \times \vec{p}$ with the kinetic energy operator $T = p^2/2m$ for a Schrödinger field of mass m in *coordinate space representation*.

(b) (6) Repeat this calculation with the same operators in *momentum space representation*.

(c) (8) Show that the expectation value of the angular momentum operator of a given wave function obeys the equation of motion

$$\frac{d}{dt} \langle L \rangle = \left\langle \vec{r} \times \vec{F} \right\rangle \tag{1}$$

where the force is $\vec{F} = -\nabla V(\vec{r})$.

[2] **Complex Potential** (20 points)

Sometimes it is useful to allow the potential energy in the time-dependent Schrödinger equation to be complex, i.e. $V(\vec{r}) = V'(\vec{r}) - iV''(\vec{r})$ where both V' and V'' are real.

(a) (17) Using the usual ansatz $\psi(\vec{r}, t) = A(\vec{r}, t) e^{\frac{i}{\hbar} S(\vec{r}, t)}$ in the Schrödinger equation with real-valued amplitude A and phase S *derive* the modified Hamilton-Jacobi equation and continutity equation for S and the particle density $\rho = A^2$ in the limit $\hbar \to 0$ in this case.

(b) (3) Discuss the differences compared to the known case of a purely real potential ($V'' = 0$). How can the additional terms involving V'' be interpreted?

[3] **Infinite Square Well with Source** (20 points)

This problem can be solved independently of problem [2].

Consider an infinite square well potential in 1 dimension with length L between $x = 0$ and $x = L$. The sqare well is given by the real part of the potential $V(x)$. In addition, inside the square well a constant imaginary part of the potential, iV_s ($V_s \in \mathbb{R}$), is acting on the particles. Hence the total potential is

$$V(x) = iV_s \text{ for } 0 \le x \le L \quad \text{and} \quad V(x) = \infty \text{ (and zero imag. part) elsewhere} \tag{2}$$

(a) (10) First consider the simple case of $V_s = 0$ (the potential is real). Write down all energy eigenvalues E_n and (properly normalized) eigenstates ψ_n in this case.

(b) (10) Now consider the general case of *non-zero* V_s. At $t = 0$ the system is prepared in a state that is the same as the ground state of problem (a), i.e. $\psi(x, 0) = \psi_0(x)$. Find the time dependent wave function of the problem by using a separation ansatz $\psi(x, t) = N(t)\psi_0(x)$.

[4] **Superposition of Two States** (25 points)

Consider a system with a discrete energy spectrum E_n, $n = 1, 2, \ldots$. Each energy eigenvalue has degeneracy 1 and the corresponding eigenstates ψ_n of the Hamilton operator H are properly normalized to unity. The system is prepared to be in a superposition (of equal weight) of the ground state and the first excited state at $t = 0$:

$$\psi(x, 0) = C\,(\psi_1 + \psi_2) \tag{3}$$

(a) (5) Write down the time-dependent wave function $\psi(x, t)$ and determine the normalization constant C so that $\psi(x, t)$ is properly normalized to unity.

(b) (10) Calculate the expectation value of the Hamilton operator (i.e. the average energy) $\langle E \rangle$.

(c) (10) Calculate the variance of the energy around its average value, $\Delta E = \left\langle \left(E - \langle E \rangle\right)^2 \right\rangle^{1/2}$

[5] **Lower Energy Bound** (15 points)

Consider a particle of mass m with a potential energy $V(\vec{r})$ which is bound from below, i.e. there is a value V_0 with $V(\vec{r}) \geq V_0$ everywhere. Let E and ψ be an eigenvalue and corresponding eigenfunction to the Hamilton operator, i.e.

$$T\psi + V\psi = E\psi \tag{4}$$

where T is the usual kinetic energy operator. Show explicitly that always $E > V_0$.

Useful Formulae

- δ-function

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{ik(x-x_0)}dk = \delta(x-x_0) \tag{5}$$

- Hamilton-Jacobi for the classical action $S(\vec{r},\vec{p},t)$

$$\frac{\partial S}{\partial t} + H(\vec{r},\vec{p}) = 0 \quad \text{with } p_i = \frac{\partial S}{\partial r_i} \tag{6}$$

- Current of the Schrödinger field

$$\vec{j}(\vec{r},t) = \frac{\hbar}{2mi}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) \tag{7}$$

- Jacobi identity

$$[F,[G,H]] + [H,[F,G]] + [G,[H,F]] = 0 \tag{8}$$

- Baker Campbell Hausdorff (if A, B commute with their commutator!)

$$e^A e^B = e^{A+B+[A,B]/2} \tag{9}$$

- Virial theorem for *stationary* states

$$2\langle T\rangle = \langle \vec{r}\cdot\nabla V\rangle \tag{10}$$

- Closure/completeness for continuous spectrum with eigenstates ψ_α

$$\int_{\text{spec}} \psi^*_\alpha(\vec{r}\,')\psi_\alpha(\vec{r})d\alpha = \delta^{(3)}(\vec{r}\,' - \vec{r}) \tag{11}$$

- Generator of Galilei boosts

$$\vec{K} = m\vec{r} - \vec{p}t \tag{12}$$

- Hermite polynomials

$$\frac{d^2}{d\xi^2}H_n(\xi) - 2\xi\frac{d}{d\xi}H_n(\xi) + 2nH_n(\xi) = 0 \tag{13}$$

$$\frac{d}{d\xi}H_n(\xi) = 2nH_{n-1}(\xi) \tag{14}$$

$$F(\xi,s) = \sum_{n\in\mathbb{N}} H_n(\xi)\frac{s^n}{n!} = e^{\xi^2-(s-\xi)^2} \tag{15}$$

- Harmonic oscillator: orthonormal energy eigenstates

$$\psi_n(x) = 2^{-\frac{n}{2}} n!^{-\frac{1}{2}}\left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) e^{-\frac{m\omega}{2\hbar}x^2} \tag{16}$$