Physics 606 — Spring 2015

Homework 7

Instructor: Rainer J. Fries *Turn in your work by April 7*

[1] **Equation of Motion for the Wigner Distribution** (30 points)

(a) Show that the Wigner distribution $W(\vec{r}, \vec{p}, t)$ of a quantum system of free particles of mass m obey the following equation

$$\frac{\partial W}{\partial t} + \frac{1}{m}\vec{p} \cdot \nabla_{\vec{r}} W = 0 \,. \tag{1}$$

(b) Now consider the special case of particles of mass m with Hamilton function $H(p, x) = p^2/2m + V(x)$ in 1 dimension. Show that in that case

$$\frac{\partial W}{\partial t} = -\big\{\big\{W, H\big\}\big\} \tag{2}$$

where the *Moyal bracket* of two functions f, g on phase space is defined as

$$\big\{\big\{f, g\big\}\big\} = \frac{2}{\hbar} f(x,p) \sin\left(\frac{\hbar}{2}\left(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x\right)\right) g(x,p) \,. \tag{3}$$

As usual the arrows over partial derivatives denote the direction in which the derivative is acting and the $\sin$-function is defined through its power series.

(c) In the situation of (b) consider the limit $\hbar \to 0$. Using the result of (b) write down the classical equation of motion and its first quantum correction (next term in powers of $\hbar$).

[2] **Scattering off a 1-D Square Potential** (25 points)

(a) Consider a potential barrier of height V_0 and width $2a$ as introduced in III.2.2 in class. Discuss energy eigenstates with energy above the barrier height, i.e. $E > V_0$. What is the general form of the energy eigenfunctions? Derive the M-matrix from the matching conditions and discuss the transmission and reflection coefficients T and R.

(b) Repeat the discussion for a potential well of depth $-V_0$ and width $2a$ as in III.3 in class. Discuss unbound energy eigenstates, i.e. $E > 0$. What is the general form of the energy eigenfunctions? Derive the M-matrix from the matching conditions and discuss the transmission and reflection coefficients T and R.
Hint: Find similarities between the situations in (a) and (b).

[3] **Half Oscillator** (20 points)

Consider a particle of mass m with potential energy

$$V(x) = \begin{cases} \infty & \text{for } x < 0 \\ \frac{1}{2}m\omega^2 x^2 & \text{for } x > 0 \end{cases} \tag{4}$$

i.e. a "halved" harmonic oscillator. Find the energy eigenvalues and properly normalized eigenfunctions for this particle.

[4] **Hamilton's Principle for Fields** (25 points)

Consider a field $\psi(x)$ as a function of coordinates $x = (x_i)_{i=1}^N$. Let $\mathcal{L}(\psi, \frac{\partial\psi}{\partial x_j}, x)$ be the *Lagrange density* for ψ, depending on ψ, its first derivatives, and the position vector x. Let

$$S[\psi] = \int_\Gamma \mathcal{L}\left(\psi, \frac{\partial\psi}{\partial x_j}, x\right) dx^N \tag{5}$$

be the action defined as an integral of the Lagrange density over a region Γ in $\mathbb{R}^N$. In the following we only consider fields ψ that take fixed values on the boundary of Γ, denoted as $\partial\Gamma$. Show that the following two statements are equivalent:[1]

(i) $\psi(x)$ is an extremum of the functional S, i.e. small variations $\delta\psi(x)$ around $\psi(x)$ consistent with the boundary conditions leave S invariant: $\delta S = 0$.

(ii) ψ satiesfies the Euler-Lagrange field equation

$$\frac{\partial\mathcal{L}}{\partial\psi} - \frac{\partial}{\partial x_j}\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial x_j}\right)} = 0. \tag{6}$$

Hint: You can parameterize small deviations from $\psi(x)$ as $\psi(x,\alpha) = \psi(x) + \alpha\eta(x)$ where α is a "small" parameter and $\eta(x)$ is a test function which has to vanish on $\partial\Gamma$. Then $\delta S = (\partial S/\partial\alpha)\delta\alpha$; OR take your favorite classical mechanics textbook, look up the derivation of the Euler-Lagrange equations from the Hamilton Principle when ψ is only a function of one parameter (time in classical mechanics) and generalize it to the case of a multi-dimensional parameter space.

[1] This statement can be easily generalized to a Lagrange density involving several fields $\psi_i(x)$, as for example required for the complex Schrödinger field.