Physics 606 — Spring 2015

Homework 6

Instructor: Rainer J. Fries *Turn in your work by March 24*

[1] **Hermite Polynomials** (25 points)

(a) Show that for Hermite polynomials of degree $n \in \mathbb{N}$

$$\frac{d}{d\xi} H_n(\xi) = 2n H_{n-1}(\xi) \tag{1}$$

(b) Prove that the generating function

$$F(\xi, s) = \sum_{n=0}^{\infty} \frac{H_n(\xi)}{n!} s^n \tag{2}$$

can be written in closed form as

$$F(\xi, s) = e^{\xi^2 - (s-\xi)^2} \tag{3}$$

Hint: Integrate the differential equation for F which follows from the relation in (a).

(c) Using the generating function show that

$$H_n(\xi) = (-1)^n e^{\xi^2} \frac{d^n}{dx^n} e^{-\xi^2} \tag{4}$$

(d) Prove the integral representation

$$H_n(\xi) = \frac{2^n}{\sqrt{\pi}} \int_{\mathbb{R}} (\xi + is)^n \, e^{-s^2} ds \, . \tag{5}$$

Hint: Induction using (a).

[2] **Important Amplitudes for the Harmonic Oscillator** (25 points)

(a) Consider the function

$$I(s, t, \lambda) = \int_{\mathbb{R}} F(\xi, s) F(\xi, t) e^{-\xi^2 + 2\lambda\xi} d\xi \tag{6}$$

which contains the generating function F for Hermite polynomials. Show with the help of problem [1] that

$$I(s, t, \lambda) = \sqrt{\pi} e^{\lambda^2 + 2(st + \lambda s + \lambda t)} \, , \tag{7}$$

and on the other hand

$$I_{nmk} := \int_{\mathbb{R}} H_n(\xi) H_m(\xi) \xi^k e^{-\xi^2} d\xi = \frac{1}{2^k} \frac{\partial^{n+m+k} I}{\partial s^n \partial t^m \partial \lambda^k} \Big|_{s,t,\lambda=0} \tag{8}$$

for $n, m, k \in \mathbb{N}$. Thus I is a generating function for integrals of the type I_{nmk}.
Hint: We have discussed the special case $k = 0$ in class.

(b) Consider a particle of mass m in a harmonic oscillator potential $\frac{1}{2}m\omega^2 x^2$. Use the results from (a) to prove that for stationary states $\psi_n(x)$

$$\langle \psi_n | x \psi_{n'} \rangle = \int_{\mathbb{R}} \psi_n(x) x \psi_{n'}(x) dx = \sqrt{\frac{\hbar}{2m\omega}} \left[\sqrt{n}\, \delta_{n',n-1} + \sqrt{n+1}\, \delta_{n',n+1} \right] . \tag{9}$$

for $n, n' \in \mathbb{N}$

(c) In the same situation as in (b), what is $\langle \psi_n | x^2 \psi_{n'} \rangle$?

[3] **Harmonic Oscillator: Averages and Virial Theorem** (25 points)

Consider a particle of mass m in a harmonic oscillator potential $\frac{1}{2}m\omega^2 x^2$.

(a) Use the explicit solution for the time evolution of the wave function $\psi(x,t)$ with given initial condition $\psi(x,0)$ at $t = 0$ to show that the expectation values of position and momentum of the particle follow a classical motion, i.e.

$$\langle x \rangle(t) = \langle x \rangle(0) \cos \omega t + \frac{\langle p \rangle(0)}{m\omega} \sin \omega t \qquad \langle p \rangle(t) = m \frac{d\langle x \rangle(t)}{dt} \tag{10}$$

Compare to the result from HW 5, [1] which was obtained without using explicit solutions to the harmonic oscillator.

(b) Calculate $\langle \psi_n | p^2 \psi_{n'} \rangle$, $n, n' \in \mathbb{N}$ for any stationary states ψ_n (p is the momentum operator). Use the result to validate the virial theorem for stationary states.

[4] **Properties of Wigner Distributions** (25 Points)

Consider the Wigner distribution $W(\vec{r}, \vec{p}, t)$ associated with a wave function $\psi(\vec{r}, t)$ as defined in class.

(a) Show that $W(\vec{r}, \vec{p}, t)$ is a real-valued function.

(b) Prove that universal upper and lower bounds for W exist, more precisely

$$-\frac{2}{h^3} \leq W(\vec{r}, \vec{p}, t) \leq \frac{2}{h^3} . \tag{11}$$

(c) Consider a second Wigner function $W'(\vec{r}, \vec{p}, t)$ which is associated with a different wave function $\psi'(\vec{r}, t)$. Show that the overlap probability of the wave functions is given by

$$|\langle \psi' | \psi \rangle|^2 = (2\pi\hbar)^3 \int W(\vec{r}, \vec{p}, t) W'(\vec{r}, \vec{p}, t) d^3r d^3p . \tag{12}$$