Physics 606 — Spring 2015

Homework 4

Instructor: Rainer J. Fries *Turn in your work by March 3*

[1] **Baker-Campbell-Hausdorff Relations** (25 points)

Let F, G be two operators on a vector space of functions.

(a) Prove that

$$e^F G e^{-F} = \sum_{k=0}^{\infty} \frac{1}{k!} \left[F, \left[F, \ldots, \left[F, G\right]\right] \ldots\right] \quad (1)$$

where the ... indicate k applications of the commutator with G.
Hint: On possible proof resolves both sides into sums of terms of the form $\sim F^{k-j} G F^j$ and uses induction.

(b) Prove the following simplified Baker-Campbell-Hausdorff formula if F and G both commute with their commutator $[F, G]$:

$$e^F e^G = e^{F+G+[F,G]/2}\,. \quad (2)$$

Hint: Consider the expression $d(e^{tF} e^{tG})/dt$ where t is a real-valued parameter. Establish a differential equation for $e^{tF} e^{tG}$ which you can easily solve.

Commutators and Poisson Brackets; Products of Conjugate Variables (25 points)

Let $H = p^2/2m + V(\vec{r})$ be the Hamilton operator of a system.

[2] (a) Calculate the commutator $[\vec{r} \cdot \vec{p}, H]$ where $\vec{r}$ and $\vec{p}$ are the position and momentum operator, respectively. Show that the commutator $[\vec{p} \cdot \vec{r}, H]$ results in the same operator.

(b) Compute $[(\vec{r} \cdot \vec{p})(\vec{r} \cdot \vec{p}), H]$ and $[(\vec{r} \cdot \vec{p})(\vec{p} \cdot \vec{r}), H]$ for the special case $V = 0$. Compare both resulting operators by applying them to a plane wave test function $e^{i(\vec{r}\cdot\vec{p}-Et)/\hbar}$.

(c) Calculate the classical Poisson bracket[1] $\{\vec{r} \cdot \vec{p}, H\}$ and compare to the result of (a). Also compare the fundamental commutators and Poisson brackets $[r_i, p_j]$ and $\{r_i, p_j\}$, $i, j = 1, 2, 3$. Does the correspondence principle between commutators of operators and Poisson brackets of their classical counterparts work in those cases?

(d) For the special case $V = 0$ calculate the classical Poisson bracket $\{(\vec{r} \cdot \vec{p})^2, H\}$ and compare to the results of (b). Is the correspondence principle tenable?

[3] **Schrödinger Equation with Electromagnetic Potential** (25 points)

Recall that the classical Hamilton function for a particle of mass m and charge q subject to electrostatic and magnetic potentials $\phi(\vec{r}, t)$ and $\vec{A}(\vec{r}, t)$, respectively, is

$$H = \frac{1}{2m} \left(\vec{p} - q\vec{A}\right)^2 + q\phi\,. \quad (3)$$

[1] We agree to use the definition $\{f, g\} = \sum_{k=1}^{n} \left(\frac{\partial f}{\partial r_k} \frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial g}{\partial r_k} \right)$.

(a) What are the Hamilton-Jacobi equation for the classical action S_{cl} and the classical continuity equation in this case?
Hint: Note that in the presence of a vector potential $\vec{A}$ the relevant velocity is $(\vec{p} - q\vec{A})/m$.

(b) Show that the *Schrödinger equation with electromagnetic potentials*

$$i\hbar \frac{\partial}{\partial t}\psi(\vec{r},t) = \frac{1}{2m}\left(-i\hbar\nabla - q\vec{A}\right)^2 \psi(\vec{r},t) + q\phi\psi(\vec{r},t) \tag{4}$$

for a wave function

$$\psi = C(\vec{r},t)e^{\frac{i}{\hbar}S(\vec{r},t)} \tag{5}$$

with real amplitude C and phase S reduces to the two classical equations from (a) for $\hbar \to 0$ and $S \to S_{\text{cl}}$.

[4] **Gauge Invariance and Lorentz Force** (25 points)

(a) Show that the Schrödinger equation from problem [3](b) is invariant under the simultaneous gauge transformations

$$\vec{A} \mapsto \vec{A} + \nabla f \qquad \phi \mapsto \phi - \frac{\partial f}{\partial t} \qquad \psi \mapsto e^{\frac{i}{\hbar}qf}\psi \tag{6}$$

where $f(\vec{r},t)$ is a real-valued function.

(b) Show that the quantum mechanical Lorentz force is given by

$$m\frac{d}{dt}\langle\vec{v}\rangle = \frac{q}{2}\left\langle \vec{v}\times\vec{B} - \vec{B}\times\vec{v}\right\rangle + q\langle\vec{E}\rangle \tag{7}$$

where the velocity operator is given by the operator identity

$$\vec{v} = \frac{1}{m}\left(\vec{p} - q\vec{A}\right) . \tag{8}$$

and $\vec{E} = -\nabla\phi - \partial\vec{A}/\partial t$ and $\vec{B} = \nabla\times\vec{A}$ are the usual electric and magnetic fields.