Physics 606 (Quantum Mechanics I) — Spring 2014

Final Exam

Instructor: Rainer J. Fries

[1] **δ-Potential** (30 points)

A particle of mass m in one dimension interacts with an attractive δ-shaped potential energy $V(x) = V_0\delta(x)$ with $V_0 < 0$.

(a) (15) Discuss the eigenvalues and eigenfunctions of the Hamilton operator $H = \frac{p^2}{2m} + V$ for particle energies $E < 0$ (i.e. bound states). How many bound states are there?

(b (15) Discuss the stationary solutions for the scattering case $E > 0$. Derive the M-matrix and the coefficients of transmission and reflexivity of the barrier for incoming plane waves.

Hint: Integrate the Schrödinger Equation in a small region around $x = 0$ to see the effect of the δ-function potential on the matching of the asymptotic solutions for $x > 0$ and $x < 0$.

[2] **Spherical Potential Well** (25 points)

Consider a particle of mass m interacting with an infinite spherical potential well of radius R in 3 dimensions, with $V(\vec{r}) = 0$ for $r < R$ and $V(\vec{r}) = \infty$ for $r > R$.

(a) (20) Write down a general ansatz for the wave function of stationary bound states and explain it. Give a set of equations that could determine all remaining parameters in your ansatz.

(b) (5) Find the energies of the lowest four energy eigenstates and describe the quantum numbers that label these states.
You do not have to find the proper normalization of those states.

[3] **Angular Momentum Algebra** (20 points)

Consider a system in an eigenstate of the squared angular momentum operator J^2 with quantum number $j = 1$. The possible eigenstates of J_z that span this space are denoted by the magnetic quantum number m as $|m\rangle = |1, m\rangle$.

(a) (10) Give the matrix elements of the operators J_x, J_y, J_z and J^2 in this J_z-representation.

(b) (5) Check the commutator $[J_x, J_y]$ explicitly in matrix representation. Does it coincide with the result you expect?

(c) (5) Calculate the expectation values $\langle J_x\rangle$ and $\langle J_x^2\rangle$ of the system using the results of (a).
If you were not able to solve (a) you can get partial credit for (c) by calculating these expectation values explicitly for the case of orbital angular momentum operators L_x and L_x^2.

[4] **Perturbation Theory** (25 points)

Consider a particle of mass m in one dimension in a harmonic oscillator potential. In addition a small constant field acts on the particle, creating a linear term $\delta V = gx$ in the Hamilton operator, i.e.

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 + \delta V \tag{1}$$

(a) (10) Using the harmonic oscillator with $g = 0$ as the unperturbed system, calculate the shift in energy for the ground state and the first excited state of the harmonic oscillator in the presence of the additional field in first order perturbation theory.

(b) (15) Since the results of (a) are quite unsatisfactory, apply the Rayleigh-Ritz method, using again just the lowest two states of the unperturbed harmonic oscillator as trial functions. Calculate the energies of the two lowest states in the presence of the field in this approximation.

Useful Formulae

- δ-function

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{ik(x-x_0)}dk = \delta(x-x_0) \tag{2}$$

- Hamilton-Jacobi for the classical action $S(\vec{r},\vec{p},t)$

$$\frac{\partial S}{\partial t} + H(\vec{r},\vec{p}) = 0 \quad \text{with } p_i = \frac{\partial S}{\partial r_i} \tag{3}$$

- Current of the Schrödinger field

$$\vec{j}(\vec{r},t) = \frac{\hbar}{2mi}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) \tag{4}$$

- Jacobi identity

$$[F,[G,H]] + [H,[F,G]] + [G,[H,F]] = 0 \tag{5}$$

- Baker Campbell Hausdorff (if A, B commute with their commutator)

$$e^A e^B = e^{A+B+[A,B]/2} \tag{6}$$

- Virial theorem for stationary states

$$2\langle T\rangle = \langle \vec{r}\cdot\nabla V\rangle \tag{7}$$

- Closure/completeness relation for orthonormal basis states $|n\rangle$

$$\sum_n |n\rangle\langle n\rangle = 1 \tag{8}$$

- Generator of Galilei boosts

$$\vec{K} = m\vec{r} - \vec{p}t \tag{9}$$

- Hermite polynomials

$$\frac{d^2}{d\xi^2}H_n(\xi) - 2\xi\frac{d}{d\xi}H_n(\xi) + 2nH_n(\xi) = 0\,, \quad n\in\mathbb{N} \tag{10}$$

$$\frac{d}{d\xi}H_n(\xi) = 2nH_{n-1}(\xi) \tag{11}$$

$$F(\xi,s) = \sum_{n\in\mathbb{N}} H_n(\xi)\frac{s^n}{n!} = e^{\xi^2-(s-\xi)^2} \tag{12}$$

- Legendre Polynomials/Functions

$$\frac{d}{d\xi}\left((1-\xi^2)\frac{d}{d\xi}\right)P_l^m(\xi) - \frac{m^2}{1-\xi^2}P_l^m(\xi) + l(l+1)P_l^m(\xi) = 0\,, \quad l\in\mathbb{N}\,, \quad -l\le m\le l \tag{13}$$

$$P_l(\xi) = P_l^0(\xi) = \frac{1}{2^l l!}\frac{d^l}{d\xi^l}(\xi^2-1)^l\,, \qquad P_l^m = (1-\xi^2)^{m/2}\frac{d^m}{d\xi^m}P_l(\xi) \tag{14}$$

- Laguerre Polynomials

$$\rho\frac{d^2}{d\rho^2}L_n^m(\rho)+(m+1+\rho)\frac{d}{d\rho}L_n^m(\rho)+nL_n^m(\rho)=0\,,\quad n\in\mathbb{N}\,,\quad n\geq 2m \tag{15}$$

$$L_n(\rho)=L_n^0(\rho)=e^\rho\frac{d^n}{d\rho^n}\left(\rho^n e^{-\rho}\right)\,,\quad L_n^m(\rho)=(-1)^m\frac{d^m}{d\rho^m}L_{n+m}(\rho) \tag{16}$$

- Spherical Bessel Functions

$$\frac{d^2}{d\rho^2}j_l(\rho)+\frac{2}{\rho}\frac{d}{d\rho}j_l(\rho)+\left(1-\frac{l(l+1)}{\rho^2}\right)j_l(\rho)=0\,,\quad l\in\mathbb{N} \tag{17}$$

$$j_l(\rho)=(-1)^l\rho^l\left(\frac{d}{\rho d\rho}\right)^l\frac{\sin\rho}{\rho} \tag{18}$$

The first zeros of j_l are (approx.) $\rho=4.49,7.73,\ldots$ ($l=1$), $\rho=5.76,9.10,\ldots$ ($l=2$), $\rho=6.99,\ldots$ ($l=3$).

- Harmonic oscillator: orthonormal energy eigenstates

$$\psi_n(x)=2^{-\frac{n}{2}}n!^{-\frac{1}{2}}\left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}}H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)e^{-\frac{m\omega}{2\hbar}x^2} \tag{19}$$

- Spherical Harmonics

$$Y_l^m(\theta,\phi)=\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-m)!}{(l+m)!}}(-1)^m P_l(\cos\theta)e^{im\phi}\,,l\in\mathbb{N}\,,\quad 0\leq m\leq l \tag{20}$$

- Angular Momentum Algebra

$$J^2|j,m\rangle=j(j+1)\hbar^2|j,m\rangle\,,\quad J_\pm|j,m\rangle=\sqrt{(j\mp m)(j\pm m+1)}\,\hbar|j,m+1\rangle \tag{21}$$

- Spherical Coordinates

$$T=-\frac{\hbar^2}{2mr^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{L^2}{2mr^2} \tag{22}$$

$$L_x=-i\hbar\left(-\sin\phi\frac{\partial}{\partial\theta}-\cos\phi\cot\theta\frac{\partial}{\partial\phi}\right) \tag{23}$$

$$L_y=-i\hbar\left(\cos\phi\frac{\partial}{\partial\theta}-\sin\phi\cot\theta\frac{\partial}{\partial\phi}\right) \tag{24}$$

$$L_z=-i\hbar\frac{\partial}{\partial\phi} \tag{25}$$