# PHYS 606 — Midterm Exam Solution — Spring 2014

[1] (a) $i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V'\psi - iV''\psi$ Schrödinger Eqn.

Ansatz $\psi(\vec{r},t) = A(\vec{r},t)\, e^{\frac{i}{\hbar}S(\vec{r},t)}$

$$\Rightarrow \quad i\hbar \frac{1}{A}\frac{\partial A}{\partial t}\psi - \frac{\partial S}{\partial t}\psi \overset{\circledast}{=} \frac{1}{2m}\left[-\hbar^2\frac{\Delta A}{A}\psi - i\hbar \nabla S\cdot\nabla A\,\frac{2}{A}\psi + (\nabla S)^2\psi - i\hbar\,\Delta S\,\psi\right] + V'\psi - iV''\psi$$

real part of $\circledast$: $\Rightarrow \quad \frac{\partial S}{\partial t} - \frac{\hbar^2}{2m}\frac{\Delta A}{A} + \frac{1}{2m}(\nabla S)^2 + V' = 0$ (1)

imaginary part of $\circledast$ $(\times \frac{1}{\hbar}2A^2)$: $\Rightarrow \quad 2A\frac{\partial A}{\partial t} + 2A\nabla A\cdot\frac{\nabla S}{m} + 2A^2\frac{\Delta S}{2m} + \frac{2}{\hbar}A^2V''$ (2)

(1) $\underset{\hbar\to 0}{\Rightarrow} \quad \frac{\partial S}{\partial t} + \frac{1}{2m}(\underbrace{\nabla S}_{\vec{p} \text{ if } S\to S_{cl}})^2 + V' = 0$ Hamilton–Jacobi; only the real part of $V$ enters

(2) $\underset{\rho = A^2}{\Rightarrow} \quad \frac{\partial \rho}{\partial t} + \underbrace{\nabla\rho\cdot\underbrace{\frac{\nabla S}{m}}_{\to\vec{v} \text{ if } S\to S_{cl}} + \rho\nabla\left(\underbrace{\frac{\nabla S}{m}}_{\to\vec{v}}\right)}_{\nabla\vec{j}_{cl}} + \frac{2\rho}{\hbar}V''$ Continuity equation with add'l term

(b) • Only the real part $V'$ enters Hamilton–Jacobi and determines the phase.

• $\frac{2\rho V''}{\hbar}$ is additional source $(V''<0)$ or sink term $(V''>0)$

(c) $i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi - iV''\psi$

Ansatz $\psi(\vec{r},t) = A(t)\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ with $\omega = \frac{k^2}{2m}$

Directly from the Schrödinger equation: $\hbar\frac{\partial A}{\partial t} = -V''A \quad \Rightarrow A = C e^{-\frac{V''}{\hbar}t}$

(or same result from modified continuity equation)

E.g. for $V''>0$: Plane waves are exponentially damped ~ absorption of particles

[2] $-\frac{\hbar^2}{2m}\Delta\phi + V\phi = E\phi$

$$V(x) = \begin{cases} \infty & \text{for } x<0 \\ \frac{1}{2}m\omega^2 x^2 & \text{for } x>0 \end{cases} \Rightarrow \phi(x) \sim \begin{cases} 0 & \text{for } x<0 \\ \psi_n(x) & \text{for } x>0 \end{cases}$$

$\underbrace{\psi_n}_{\text{"full"}}$ harm. osc. solutions

$\psi(x)=0$ @ $V(x)=\infty$ clear from finiteness of energy

Matching condition from continuity equation $\psi(x\to 0^-) = \psi(x\to 0^+)$

$\Rightarrow$ only $\psi_n(x)$ with $\psi_n(x)=0$ allowed $\Rightarrow$ <u>odd $n$</u> solutions.

$\Rightarrow$ Energy eigenvalues $E_n = (n+\frac{1}{2})\hbar\omega$ with $n = 1, 3, 5, \dots$

i.e. $E_1 = \frac{3}{2}\hbar\omega$, $E_3 = \frac{7}{2}\hbar\omega, \dots$

Eigenfcts. $$\phi_n(x) = \begin{cases} 0 & \text{for } x<0 \\ 2^{-\frac{n}{2}+1}\frac{1}{\sqrt{n!}}\left(\frac{m\omega}{\hbar\pi}\right)^{1/4} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)e^{-\frac{m\omega}{2\hbar}x^2} & \text{for } x>0 \end{cases}$$

Normalization adjustments: $\int_{\mathbb{R}} |\phi_n|^2 dx = \int_0^\infty \underbrace{|\psi_n|^2}_{\text{even fct. in } x} dx = \frac{1}{2}\int_0^\infty |\psi_n|^2 dx = \frac{1}{2}$

$\Rightarrow$ add'l factor $\sqrt{2}$ needed for proper normalization

[3] (a) $\frac{d}{dt}\langle r_i\rangle = \frac{1}{i\hbar}\langle [r_i, H]\rangle = \frac{1}{i\hbar}\langle [r_i, \frac{p^2}{2m}]\rangle = \frac{1}{i\hbar}\sum_j \frac{1}{2m}\langle p_j \underbrace{[r_i, p_j]}_{i\hbar\delta_{ij}} + \underbrace{[r_i, p_j]}p_j\rangle$

$= \frac{\langle p_i\rangle}{m}$

$\frac{d}{dt}\langle p_i\rangle = \frac{1}{i\hbar}\langle [p_i, H]\rangle = \frac{1}{i\hbar}\langle [-i\hbar\frac{\partial}{\partial r_i}, V(\vec r)]\rangle = -\langle \frac{\partial}{\partial r_i}V\rangle$

$\Rightarrow m\frac{d^2}{dt^2}\langle \vec r\rangle = -\langle \nabla V(\vec r)\rangle$

(b) Slowly varying $V(\vec r)$: Taylor expand around $V(\langle \vec r\rangle)$.

$V(\vec r) = V(\langle\vec r\rangle) + (r_i - \langle r_i\rangle)\frac{\partial V}{\partial r_i}(\langle \vec r\rangle) + \frac{1}{2!}(r_i - \langle r_i\rangle)(r_j - \langle r_j\rangle)\frac{\partial^2 V}{\partial r_i \partial r_j}(\langle\vec r\rangle)$

$$+\frac{1}{3!}(r_i-\langle r_i\rangle)(r_j-\langle r_j\rangle)(r_k-\langle r_k\rangle)\frac{\partial^3 V}{\partial r_i\partial r_j\partial r_k}+\dots$$

$$\left\langle \frac{\partial}{\partial r_e}V(\vec r)\right\rangle = \frac{\partial V}{\partial r_e}(\langle\vec r\rangle) + \underbrace{\langle(r_i-\langle r_i\rangle)\rangle}_{=0}\frac{\partial^2 V}{\partial r_i\partial r_e}(\langle\vec r\rangle)+\frac{1}{2!}\underbrace{\langle(r_i-\langle r_i\rangle)(r_j-\langle r_j\rangle)\rangle}_{\substack{=0 \text{ for } i\neq j\\ =(\Delta r_i)^2 \text{ for } i=j}}\frac{\partial^3 V(\langle\vec r\rangle)}{\partial r_i\partial r_j\partial r_e}$$

$$=\frac{\partial V(\langle\vec r\rangle)}{\partial r_e}+\frac{1}{2!}(\Delta\vec r)^2\frac{\partial}{\partial r_e}\Delta V(\langle\vec r\rangle)$$

$$\Rightarrow m\frac{d^2}{dt^2}\langle\vec r\rangle = -\nabla\Big[V(\langle\vec r\rangle)+\frac{1}{2!}(\Delta\vec r)^2\Delta V(\langle\vec r\rangle)\Big]$$

Newton's Law

1st quantum correction vanishes for width $(\Delta\vec r)^2$ of wave packet $\to 0$

[4] (a) $\frac{d}{dt}\langle p^n\rangle = \frac{1}{i\hbar}\langle[p^n,\frac{p^2}{2m}]\rangle = 0 \quad\Rightarrow\quad \langle p\rangle = \text{const} = \langle p\rangle(t=0)$

any $n\in\mathbb{N}$ — free particle

$\langle p^2\rangle = \text{const} = \langle p^2\rangle(t=0)$

$\langle p^3\rangle = \text{const} = \langle p^3\rangle(t=0)$

$$\Rightarrow \gamma_1^p = \frac{\langle(p-\langle p\rangle)^3\rangle}{\langle(p-\langle p\rangle)^2\rangle^{3/2}} = \text{const} = \gamma_1^p(0)$$

(b) $\frac{d}{dt}\langle x^3\rangle = \frac{1}{i\hbar}\langle[x^3,\frac{p^2}{2m}]\rangle = \frac{1}{i\hbar}\langle x^2\underbrace{[x,\frac{p^2}{2m}]}_{i\hbar\frac{p}{m}}+x[x,\frac{p^2}{2m}]x+[x,\frac{p^2}{2m}]x^2\rangle$

$$=\frac{1}{m}\langle x^2p+xpx+px^2\rangle$$

$$\frac{d^2}{dt^2}\langle x^3\rangle = \frac{1}{m}\frac{1}{i\hbar}\langle[x^2p,\frac{p^2}{2m}]+[xpx,\frac{p^2}{2m}]+[px^2,\frac{p^2}{2m}]\rangle$$

$$=\frac{(i\hbar)^{-1}}{2m^2}\langle x(2i\hbar p)p+(2i\hbar p)xp+(2i\hbar p)px+xp(2i\hbar p)+px(2i\hbar p)+p(2i\hbar p)x\rangle$$

$$=\frac{1}{m^2}\langle 2xp^2+\underbrace{pxp}_{xp^2-i\hbar p}+\underbrace{pxp}_{p^2x+i\hbar p}+2p^2x\rangle = \frac{3}{m^2}\langle xp^2+p^2x\rangle$$

$$\frac{d^3}{dt^3}\langle x^3\rangle = \frac{3}{m^2}\frac{1}{i\hbar}\langle[xp^2,\frac{p^2}{2m}]+[p^2x,\frac{p^2}{2m}]\rangle = \frac{3}{m^2}\frac{1}{i\hbar}\langle\frac{i\hbar p^3}{m}+\frac{i\hbar p^3}{m}\rangle = \frac{6}{m^3}\langle p^3\rangle$$

$$\Rightarrow \frac{d^3}{dt^3}\langle x^3\rangle = \frac{6}{m^3}\langle p^3\rangle(0) = \text{const.}$$