

PHYS 606 - Spring 2014 - HW VII Solution

[1] (a) Use power series

$$\begin{aligned} \frac{dH_n}{d\xi} &\stackrel{\uparrow}{=} \frac{d}{d\xi} \left[ (-1)^{\frac{n}{2}} \frac{n!}{(\frac{n}{2})!} \left( 1 - \frac{2n}{2!} \xi^2 + \frac{2^2 n(n-2)}{4!} \xi^4 - \dots \right) \right] \\ &\stackrel{\text{even}}{=} (-1)^{\frac{n}{2}} \frac{n!}{(\frac{n}{2})!} \left( -\frac{2n}{2!} \xi + \frac{2^2 n(n-2)}{3!} \xi^3 - \dots \right) \\ &= -2n (-1)^{\frac{n}{2}} \frac{n}{2} \frac{(n-1)!}{(\frac{n}{2}-1)!} \left( \xi - \frac{2(n-2)}{3!} \xi^3 + \dots \right) \\ &= 2n (-1)^{\frac{n-2}{2}} \frac{d(n-1)!}{(\frac{n-2}{2})!} \left( \xi - \frac{2(n-2)}{3!} \xi^3 + \dots \right) = 2n H_{n-1}(\xi) \end{aligned}$$

(n-1) odd

$$\begin{aligned} \frac{dH_n}{d\xi} &\stackrel{\uparrow}{=} \frac{d}{d\xi} \left[ (-1)^{\frac{n-1}{2}} \frac{2n!}{(\frac{n-1}{2})!} \left( \xi - \frac{2(n-1)}{3!} \xi^3 + \frac{2^2 (n-1)(n-3)}{5!} \xi^5 - \dots \right) \right] \\ &\stackrel{\text{odd}}{=} 2n (-1)^{\frac{n-1}{2}} \frac{(n-1)!}{(\frac{n-1}{2})!} \left( 1 - \frac{2(n-1)}{2!} \xi^2 + \frac{2^2 (n-1)(n-3)}{4!} \xi^4 - \dots \right) \\ &= 2n H_{n-1}(\xi) \end{aligned}$$

(n-1) even

(b)  $\frac{d}{d\xi} F(\xi, s) = \sum_{n=0}^{\infty} \frac{dH_n}{d\xi} \frac{s^n}{n!} = 2 \sum_{n=0}^{\infty} \frac{H_{n-1}}{(n-1)!} s^n = 2s \sum_{n=0}^{\infty} \frac{H_n}{n!} s^n = 2s F(\xi, s)$

$\Rightarrow F(\xi, s) = C e^{2\xi s}$

and  $F(0, s) = \sum_{n=0}^{\infty} \frac{H_n(0)}{n!} s^n = \sum_{\substack{n=0 \\ \text{even}}}^{\infty} (-1)^{\frac{n}{2}} \frac{n!}{(\frac{n}{2})!} \frac{s^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{s^{2n}}{n!} = e^{-s^2} \equiv C'$

only constant term of even n contribute      |      n → 2n

$\Rightarrow F(\xi, s) = e^{-s^2 + 2\xi s} = e^{\xi^2 - (s-\xi)^2}$

(c)  $H_n(\xi) = \frac{d^n}{ds^n} F(\xi, s) \Big|_{s=0} = e^{\xi^2} \frac{d^n}{ds^n} e^{-(s-\xi)^2} \Big|_{s=0} = (-1)^n e^{\xi^2} \frac{d^n}{ds^n} e^{-(s-\xi)^2} \Big|_{s=0} = (-1)^n e^{\xi^2} \frac{d^n}{ds^n} e^{-s^2}$

(d) Induction:  $\frac{d}{d\xi} \frac{2^n}{\sqrt{\pi}} \int_{\mathbb{R}} (\xi + is)^n e^{-s^2} ds = 2n \frac{2^{n-1}}{\sqrt{\pi}} \int_{\mathbb{R}} (\xi + is)^{n-1} e^{-s^2} ds$

same as the  $H_n(\xi)$ , see (a); and

$H_0(\xi) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} 1 \cdot e^{-s^2} ds = 1 \quad \checkmark$

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$$[2] (a) I = \int_{\mathbb{R}} F(\xi, s) F(\xi, t) e^{-\xi^2 + 2\lambda\xi} d\xi =$$

$$(a) \int_{\mathbb{R}} e^{\xi^2 - (s-\xi)^2 + \xi^2 - (t-\xi)^2} e^{-\xi^2 + 2\lambda\xi} d\xi$$

$$= e^{\lambda^2 + 2(st + \lambda s + \lambda t)} \int_{\mathbb{R}} e^{-(\xi - s - t - \lambda)^2} d\xi = \sqrt{\pi} e^{\lambda^2 + 2(st + \lambda s + \lambda t)}$$

On the other hand  $I = \sum_{n, m, k=0}^{\infty} \int_{\mathbb{R}} H_n(\xi) \frac{s^n}{n!} H_m(\xi) \frac{t^m}{m!} e^{-\xi^2} \xi^k \frac{(2\lambda)^k}{k!} d\xi$

$$\Rightarrow I_{nmk} = \int_{\mathbb{R}} H_n(\xi) H_m(\xi) e^{-\xi^2} \xi^k d\xi = \frac{1}{2^k} \left. \frac{\partial^{n+m+k} I}{\partial s^n \partial t^m \partial \lambda^k} \right|_{s, t, \lambda=0}$$

$$(b) \langle \psi_n | x \psi_{n'} \rangle = \int_{\mathbb{R}} \psi_n(x) x \psi_{n'}(x) dx = 2^{-\frac{1}{2}(n+n')} \frac{1}{\sqrt{n!n'}} \sqrt{\frac{\hbar}{m\omega}} \int_{\mathbb{R}} H_n(\xi) H_{n'}(\xi) e^{-\xi^2} \xi d\xi$$

$$(a) 2^{-\frac{1}{2}(n+n')} \frac{1}{\sqrt{n!n'}} \sqrt{\frac{\hbar}{m\omega}} I_{nn'1}$$

$$\xi = \sqrt{\frac{m\omega}{\hbar}} x$$

$$(a) 2^{-\frac{1}{2}(n+n')} \frac{1}{\sqrt{n!n'}} \sqrt{\frac{\hbar}{m\omega}} \frac{1}{2} \left. \frac{\partial^{n+n'+1}}{\partial s^n \partial t^{n'}} \right|_{s, t=0} [2(st) e^{2st}]$$

$$\longrightarrow 2 \sum_{l=0}^{\infty} \left[ \frac{s^{2l+1} (2t)^l}{e!} + \frac{t^{2l+1} (2s)^l}{e!} \right]$$

$$= 2^{-\frac{1}{2}(n+n')} \frac{1}{\sqrt{n!n'}} \sqrt{\frac{\hbar}{m\omega}} \sum_{l=0}^{\infty} \left[ \delta_{n, 2l+1} \delta_{n', 2l} n! 2^{2l} + \delta_{n, 2l} \delta_{n', 2l+1} n'! 2^{2l} \right]$$

$$= \sqrt{\frac{\hbar}{m\omega}} \left[ \delta_{n, n'+1} \sqrt{n} 2^{-\frac{1}{2}} + \delta_{n, n'+1} \sqrt{n'} 2^{-\frac{1}{2}} \right]$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \left[ \delta_{n, n'-1} \sqrt{n} + \delta_{n, n'+1} \sqrt{n+1} \right]$$

$$(c) \langle \psi_n | x^2 \psi_{n'} \rangle = 2^{-\frac{1}{2}(n+n')} \frac{1}{\sqrt{n!n'}} \frac{\hbar}{m\omega} \frac{1}{\sqrt{\pi}} I_{nn'2}$$

$$= 2^{-\frac{1}{2}(n+n')} \frac{1}{\sqrt{n!n'}} \frac{\hbar}{m\omega} \frac{1}{4} \left. \frac{\partial^{n+n'+2}}{\partial s^n \partial t^{n'}} \right|_{s, t=0} \left[ (2 + 4(st))^2 e^{2st} \right]$$

$$2 \sum_{l=0}^{\infty} \left[ \frac{(2s)^l + 2(2st)^{2l}}{e!} + \frac{2s^{2l+2} (2t)^l}{e!} + \frac{2t^{2l+2} (2s)^l}{e!} \right]$$

$$\begin{aligned}
 &= 2^{-\frac{i}{\hbar}(n+n')t} \frac{\hbar}{m\omega} \frac{1}{2} \sum_{l=0}^{\infty} \left[ \delta_{n,l} \delta_{n',l} 2^n n! + \delta_{n,l+1} \delta_{n',l+1} 2^{n+1} n n! + \delta_{n,l+2} \delta_{n',l} 2^{n+1} n! \right. \\
 &\quad \left. + \delta_{n,l+2} \delta_{n',l} 2^{n+1} n! \right] \\
 &= \frac{\hbar}{2m\omega} \left[ \delta_{n,n'} + \delta_{n,n'} 2n + \delta_{n,n'+2} \sqrt{n(n-1)} + \delta_{n',n+2} \sqrt{n'(n-1)} \right] \\
 &= \frac{\hbar}{2m\omega} \left[ (2n+1) \delta_{n,n'} + \delta_{n',n-2} \sqrt{n(n-1)} + \delta_{n',n+2} \sqrt{(n+2)(n-1)} \right]
 \end{aligned}$$

[3] Let  $\psi(x,0) = \sum_{n \in \mathbb{N}} c_n \psi_n(x)$  with  $\psi_n$  energy eigenstates

$$\psi(x,t) = \sum_{n \in \mathbb{N}} c_n \psi_n(x) e^{-\frac{i}{\hbar}(n+\frac{1}{2})\hbar\omega t}$$

$$\langle x \rangle = \langle \psi(x,t) | x | \psi(x,t) \rangle = \sum_{n,n' \in \mathbb{N}} c_n^* c_{n'} e^{-i(n'-n)\omega t} \langle \psi_n | x | \psi_{n'} \rangle$$

$$\stackrel{(2.3.1b)}{=} \sum_{n,n'} c_n^* c_{n'} e^{-i(n'-n)\omega t} \sqrt{\frac{\hbar}{2m\omega}} \left[ \delta_{n',n-1} \sqrt{n} + \delta_{n',n+1} \sqrt{n+1} \right]$$

$$= \sum_{n \in \mathbb{N}} \left[ c_n^* c_{n-1} e^{i\omega t} \sqrt{n} + c_{n+1}^* c_n \sqrt{n+1} e^{-i\omega t} \right] \sqrt{\frac{\hbar}{2m\omega}}$$

Write  $c_i = |c_i| e^{i\phi_i} \quad \forall i \in \mathbb{N}$

$$\Rightarrow \langle x \rangle = \sum_{n \in \mathbb{N}} \sqrt{n} |c_n| |c_{n-1}| \left( e^{i(\omega t + \phi_{n-1} - \phi_n)} + e^{-i(\omega t + \phi_{n-1} - \phi_n)} \right) \sqrt{\frac{\hbar}{2m\omega}} \quad (*)$$

$2 \cos(\omega t + \phi_{n-1} - \phi_n)$

Obviously  $\langle x \rangle(t=0) = \sqrt{\frac{2\hbar}{2m\omega}} \sum_{n \in \mathbb{N}} \sqrt{n} |c_n| |c_{n-1}| \cos(\phi_{n-1} - \phi_n) \equiv \langle x \rangle_0$

Momentum op:  $p \psi_n(x) = -i\hbar \sqrt{\frac{m\omega}{\hbar}} \frac{d}{d\xi} C_n H_n(\xi) e^{-\xi^2/2} = -i\hbar \sqrt{\frac{m\omega}{\hbar}} C_n (2n H_{n-1}(\xi) - \xi H_n(\xi)) e^{-\xi^2/2}$

$$= i\hbar \frac{m\omega}{\hbar} x \psi_n(x) - i\hbar \sqrt{\frac{m\omega}{\hbar}} \sqrt{2n} \psi_{n-1}(x)$$

$$\Rightarrow \langle \psi_n | p | \psi_n \rangle = i\hbar \sqrt{\frac{m\omega}{\hbar}} \left[ \delta_{n,n-1} \sqrt{\frac{n}{2}} - \delta_{n,n-1} \sqrt{\frac{n}{2}} \right]$$

$$\langle p \rangle = \langle \psi(x,t) | p | \psi(x,t) \rangle = i\hbar \sqrt{\frac{m\omega}{\hbar}} \sum_{n,n'} c_n^* c_{n'} e^{-i(n'-n)\omega t} \left[ \delta_{n',n-1} \sqrt{\frac{n}{2}} - \delta_{n',n-1} \sqrt{\frac{n'}{2}} \right]$$

$$= i\hbar \sqrt{\frac{m\omega}{\hbar}} \sum_{n \in \mathbb{N}} \left[ c_n^* c_{n-1} e^{i\omega t} \sqrt{\frac{n}{2}} - c_{n-1}^* c_n e^{-i\omega t} \sqrt{\frac{n'}{2}} \right]$$

$$= -\sqrt{2\hbar m\omega} \sum_{n \in \mathbb{N}} \sqrt{n} |c_n| |c_{n-1}| \sin(\omega t + \phi_{n-1} - \phi_n)$$

$$\Rightarrow \langle p \rangle(t) = -\sqrt{2m\hbar\omega} \sum_{n \in \mathbb{N}} \sqrt{n} |c_n| |c_{n-1}| \sin(\psi_{n-1} - \psi_n) \equiv \langle p \rangle_0$$

and  $\langle p \rangle = m \frac{d}{dt} \langle x \rangle$  from (\*)

also from (\*):  $\langle x \rangle = \langle x \rangle_0 \cos \omega t + \frac{\langle p \rangle_0}{m\omega} \sin \omega t$

Same as in HW I, [1]

(b)  $p^2 = 2m(H - V) = 2mH - m^2\omega^2 x^2$  as operators

$$\Rightarrow \langle \psi_n | p^2 \psi_{n'} \rangle = 2m \langle \psi_n | H \psi_{n'} \rangle - m^2\omega^2 \langle \psi_n | x^2 \psi_{n'} \rangle$$

$$\Rightarrow \langle p^2 \rangle = \langle \psi_n | p^2 \psi_n \rangle = \underbrace{2m\hbar\omega(n + \frac{1}{2})}_{2m\langle H \rangle} - \underbrace{\hbar m\omega \frac{1}{2}(2n+1)}_{2m\langle V \rangle} = m\hbar\omega(n + \frac{1}{2}) \quad \text{for diagonal case } n=n'$$

In particular  $\langle T \rangle = \langle V \rangle$

Virial theorem:  $\langle T \rangle = \frac{1}{2} \langle x \frac{dV}{dx} \rangle = \langle V \rangle$  for harm. osc. ✓

For case  $n \neq n'$ :  $\langle \psi_n | p^2 \psi_{n'} \rangle = -m^2\omega^2 \langle \psi_n | x^2 \psi_{n'} \rangle$ , cf. [2](c)

[4] (a)  $E > V_0 \Rightarrow$  plane waves everywhere. General solution

$$\psi(x) = \begin{cases} A e^{ikx} + B e^{-ikx} & \text{f. } x < -a \\ C e^{ik'x} + D e^{-ik'x} & \text{f. } -a < x < a \\ E e^{ikx} + F e^{-ikx} & \text{f. } x > a \end{cases} \quad \begin{matrix} k = \frac{1}{\hbar} \sqrt{2mE} \\ k' = \frac{1}{\hbar} \sqrt{2m(E - V_0)} \end{matrix}$$

Matching: @  $x = -a$ :  $A e^{-ika} + B e^{+ika} = C e^{-ik'a} + D e^{+ik'a}$  (1)

$$ik(A e^{-ika} - B e^{+ika}) = ik'(C e^{-ik'a} - D e^{+ik'a}) \quad (2)$$

$$\Rightarrow ik \textcircled{1} + \textcircled{2}: A = \frac{1}{2k} e^{ika} \left[ C(k+k') e^{-ik'a} + D(k-k') e^{ik'a} \right] \quad (1')$$

$$ik \textcircled{1} - \textcircled{2}: B = \frac{1}{2k} e^{-ika} \left[ C(k-k') e^{-ik'a} + D(k+k') e^{ik'a} \right] \quad (2')$$