PHYS 606 – Spring 2014 – HW VI Solution

[1] (a) Eigenvalues and eigenfcts. for mom. operator p are usually $K \in \mathbb{R}$ and $\sim e^{\frac{i}{\hbar}Kx}$. Since p and U_a commute try the same eigenfcts:

$$U_a e^{\frac{i}{\hbar}Kx} = \sum_{j=0}^{\infty} \left(\frac{i}{\hbar} p a\right)^j e^{\frac{i}{\hbar}Kx} = e^{\frac{i}{\hbar}Ka} \; e^{\frac{i}{\hbar}Kx}$$

$\Rightarrow$ eigenvalues are $e^{\frac{i}{\hbar}Ka}$ with eigenfct. $\sim e^{\frac{i}{\hbar}Kx}$

But eigenvalues $e^{\frac{i}{\hbar}Ka}, e^{\frac{i}{\hbar}K'a}$ with $K-K' = \frac{2\pi\hbar}{a}$ are actually the same

$\Rightarrow$ The set $e^{\frac{i}{\hbar}Ka}$ with $-\frac{h}{2a} < K < \frac{h}{2a}$ is a unique set of eigenvalues, each with an infinite but countable degeneracy. As a set it covers the unit circle in $\mathbb{C}$.

The fcts. $e^{\frac{i}{\hbar}(K+\frac{2\pi\hbar n}{a})x}$ for $n \in \mathbb{Z}$ have all the same eigenvalue $e^{\frac{i}{\hbar}Ka}$, and they are mutually orthogonal.

(b) Fourier's Theorem: the fcts. $e^{i2\pi n\frac{x}{a}}$ $n \in \mathbb{Z}$ are periodic with period a. They are a complete basis spanning the space of square-integrable periodic fcts with period a.

Thus: arbitrary fct. in eigenspace of eigenvalue $e^{\frac{i}{\hbar}Ka}$ is a linear combination

$$\sum_{n\in\mathbb{Z}} c_n e^{\frac{i}{\hbar}(K+\frac{2\pi\hbar n}{a})x} = e^{\frac{i}{\hbar}Kx} \underbrace{\sum_{n\in\mathbb{Z}} c_n e^{i2\pi n\frac{x}{a}}}_{\text{Fourier series}} = e^{\frac{i}{\hbar}Kx} \underbrace{u(x)}_{a\text{-periodic fct.}}$$

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[2] (a) $$\mathcal{D}_{w_i} = e^{\frac{i}{\hbar}(mr_i - p_i t)w_i} = e^{\frac{i}{\hbar}(mr_i - p_i t)w_i + [\frac{i}{\hbar}mr_i w_i, -\frac{i}{\hbar}p_i w_i t]/2} \, e^{-\frac{i}{2\hbar}mw_i^2 t}$$

$\frac{i}{2\hbar}mw_i^2 t$

exponent just a number, so can be pulled out

$$\underset{BCH}{=} e^{\frac{i}{\hbar}mr_i w_i} e^{-\frac{i}{\hbar}p_i w_i t} e^{-\frac{i}{\hbar}\frac{mw_i^2}{2}t}$$

just numbers

commutes here

$$\Rightarrow \mathcal{D}_{w_i}\psi(\vec{r},t) = e^{\frac{i}{\hbar}(mw_i r_i - \frac{mw_i^2}{2}t)} \underbrace{e^{-\frac{i}{\hbar}p_i w_i t}}_{\text{translation operator by } -\vec{w}t} \psi(\vec{r},t) = e^{\frac{i}{\hbar}(mw_i r_i - \frac{mw_i^2}{2}t)}\psi(\vec{r}-\vec{w}t,t)$$

(b) $$[K_i, K_j] = [mr_i - p_i t, mr_j - p_j t] = 0$$

$$[K_i, p_j] = [mr_i, p_j] = i\hbar m\,\delta_{ij}$$

$$[K_i, H] = [mr_i, T] \underset{I.8.2}{=} i\hbar p_i$$

(c) $$e^{\frac{i}{\hbar}\vec{K}\vec{w}_2} e^{\frac{i}{\hbar}\vec{K}\vec{w}_1} \underset{(b)}{=} e^{\frac{i}{\hbar}\vec{K}(\vec{w}_2+\vec{w}_1)}$$

$$e^{\frac{i}{\hbar}\vec{K}\cdot\vec{w}_1} e^{\frac{i}{\hbar}\vec{p}\cdot\vec{a}} = e^{\frac{i}{\hbar}(\vec{K}\cdot\vec{w}_1 + \vec{p}\cdot\vec{a} + \frac{i}{\hbar}w_{1j}\overbrace{[K_j, p_\ell]}^{i\hbar m\delta_{j\ell}}a_\ell/2)}$$

$$= \underbrace{e^{-\frac{i}{\hbar}\frac{m}{2}\vec{w}\cdot\vec{a}}}_{} e^{\frac{i}{\hbar}(\vec{K}\cdot\vec{w}_1 + \vec{p}\cdot\vec{a})}$$

appearance of this phase factor means this is a projective representation.

[3] (a) Simultaneous momentum eigenstates: separation $\psi(\vec{r}) = X(x)\,Y(y)\,Z(z)$

$$-i\hbar\frac{\partial}{\partial x}\psi(x,y,z) = p_x\psi(x,y,z) \quad\Rightarrow\quad X(x) = \text{const} * e^{\frac{i}{\hbar}p_x x}, \quad p_x \in \mathbb{R}$$

p_x ↑ eigenvalue; (p_x Hermitian op.)

same for y, z-directions

$$\Rightarrow \psi(x,y,z) = \text{const.} * e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}$$ eigenfct. for p_x, p_y, p_z simultaneously

with $(p_x, p_y, p_z) \in \mathbb{R}^3$

Instead of the separation ansatz you can also use our results for free particles

Impose boundary conditions: for periodic boundary cond. on box of size L the fct. $\psi(\vec{r})$ needs to be periodic w/ period L:

$$\psi(x,y,z) = \psi(x+n_1L,\ y+n_2L,\ z+n_3L) \qquad \text{with } (n_1,n_2,n_3) \in \mathbb{Z}^3$$

$$\Rightarrow \frac{i}{\hbar} p_i L = 2\pi i n_i \Rightarrow p_i = \frac{2\pi\hbar}{L} n_i = \frac{h}{L} n_i$$

$\Rightarrow$ eigenvalues that obey B.c. are $\vec{p}_N = \frac{h}{L} \underbrace{(n_1, n_2, n_3)}_{=N \in \mathbb{Z}^3}$

with eigenfct. $\psi_N(\vec{r}) = L^{-3/2} e^{\frac{i}{\hbar}\vec{p}_N \cdot \vec{r}}$

normalization from $\int_{Box} |\psi_N|^2 d^3r = 1$

$$H\psi_N(\vec{r}) = \frac{p_N^2}{2m}\psi_N(\vec{r}) \Rightarrow \psi_N \text{ are eigenstate of } H \text{ with eigenvalues}$$

$$E_N = \frac{h^2}{2mL^2}(n_1^2 + n_2^2 + n_3^2) \geq 0$$

For $L \to \infty$ the lattice of the $\vec{p}_N = \frac{h}{L} N$ has lattice spacing $\frac{h}{L} \to 0$

i.e it will cover all of $\mathbb{R}^3$ with eigenfct. $\sim e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}$ (as in I. II. 4, case (B))

(b) Number of states in a $\Delta n_x \Delta n_y \Delta n_z$ cube of the lattice is $\Delta n_x \Delta n_y \Delta n_z = \Delta N$.

Its momentum space volume is $\Delta p_x \Delta p_y \Delta p_z = \frac{h^3}{L^3} \Delta n_x \Delta n_y \Delta n_z$

$$\Rightarrow \rho = \frac{\Delta N}{L^3 \Delta p_x \Delta p_y \Delta p_z} = \frac{1}{h^3}$$

, i.e. we have one eigenstate per elementary phase space volume h^3

(c) An energy interval ΔE cuts out a spherical shell in the $\mathbb{Z}^3$ lattice of momentum eigenvalues.

The number of eigenvalues in that shell grows quadratically with its radius.

For large E (large radius) we should be able to approximate the sum over points in the shell by an integral over the shell

Number of eigenstates $\Delta N \approx \underbrace{\rho}_{\text{phase space density}} L^3 \underbrace{4\pi p^2}_{\text{shell volume in } \vec{p}\text{-space}} \Delta p$ for large p but $\Delta p \ll p$

$p^2 = 2mE \Rightarrow \Delta p = \frac{m\,\Delta E}{p}$

$$\Rightarrow \sigma = \frac{\Delta N}{\Delta E} = \frac{L^3}{h^3} 4\pi \sqrt{2mE}\, m = \frac{m^{3/2} L^3}{\sqrt{2}\,\pi^2 \hbar^3}\sqrt{E}$$

[4] (a) Cylindrical coordinates: $\begin{pmatrix} r = \sqrt{x^2+y^2} \\ \phi = \arctan\frac{y}{x} \\ z = z \end{pmatrix}$

Unit vectors:
$\hat{r} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$
$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$

Jacobi matrix

$$J = \frac{\partial(r,\phi,z)}{\partial(x,y,z)} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ \frac{\partial \phi}{\partial x} & \cdots & \\ \vdots & & \end{pmatrix} = \begin{pmatrix} \frac{x}{r} & \frac{y}{r} & 0 \\ \frac{-y/x^2}{1+y^2/x^2} & \frac{1/x}{1+y^2/x^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\frac{1}{r}\sin\phi & \frac{1}{r}\cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\nabla = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} = J^T \begin{pmatrix} \partial/\partial r \\ \partial/\partial \phi \\ \partial/\partial z \end{pmatrix} = \begin{pmatrix} \cos\phi\frac{\partial}{\partial r} - \frac{1}{r}\sin\phi\frac{\partial}{\partial\phi} \\ \sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\phi\frac{\partial}{\partial\phi} \\ \frac{\partial}{\partial z} \end{pmatrix} = \hat{r}\frac{\partial}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial\phi} + \hat{z}\frac{\partial}{\partial z}$$

$$\Delta = \nabla^2 = \frac{\partial^2}{\partial r^2} + \underbrace{\hat{\phi}\cdot\left(\frac{\partial}{\partial\phi}\hat{r}\right)}_{\hat{\phi}}\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\underbrace{}_{+\frac{\partial^2}{\partial z^2}} = \underbrace{\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}}_{\text{often written as } \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}$$

(b) Problem with cylindrical symmetry with $r = R$ fixed.

$$H = -\frac{\hbar^2}{2m}\Delta = -\frac{\hbar^2}{2m}\left(\frac{1}{R^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\right), \quad \text{solve } H\psi = E\psi$$

Separation ansatz $\psi(\phi,z) = C\,\Phi(\phi)\,Z(z) \Rightarrow \begin{cases} \frac{\partial^2 Z}{\partial z^2} = -k_z^2 Z \\ \frac{1}{R^2}\frac{\partial^2}{\partial\phi^2}\Phi = -k_\phi^2\Phi \\ \text{and } \frac{\hbar^2}{2m}(k_\phi^2 + k_z^2) = E \end{cases}$

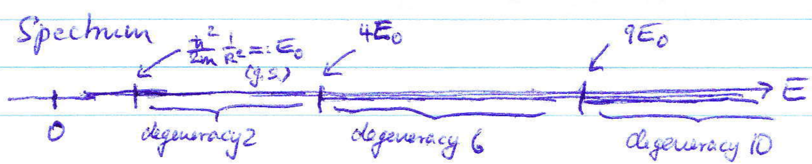
$\Rightarrow Z_{k_z}(z) = e^{ik_z z}$, $k_z \in \mathbb{R}$ free motion in z-direction

$\Rightarrow \Phi_{k_\phi}(\phi) = e^{iRk_\phi\phi}$, $k_\phi \in \mathbb{R}$

Periodicity of Φ: $\Phi(\phi) = \Phi(\phi + 2\pi n) \Rightarrow 2\pi R k_\phi \;\; 2\pi n \Rightarrow k_\phi = \frac{n}{R} \quad n \in \mathbb{Z}$

$$\Rightarrow E_{n,k_z} = \frac{\hbar^2}{2m}\left(\frac{n^2}{R^2} + k_z^2\right), \; n \in \mathbb{Z}, k_\phi \in \mathbb{R} \text{ and } \psi_{n,k_z} = C e^{in\phi} e^{ik_z z}$$

Spectrum



Possible normalization:

$$C = \underbrace{\frac{1}{\sqrt{2\pi\hbar}}}_{\text{free plane wave normalization}} \underbrace{\frac{1}{\sqrt{2\pi R}}}_{\text{azimuthal part normalized to 1}} = \frac{1}{2\pi\sqrt{R\hbar}}$$