PHYS 606 — Spring 2014 — HW V Solution

[1] (a) $\frac{d}{dt}\langle x\rangle = \frac{1}{i\hbar}\langle [x,H]\rangle = \frac{1}{i\hbar}\left\langle \frac{1}{2m}\, 2i\hbar p\right\rangle = \frac{\langle p\rangle}{m}$

$\frac{d}{dt}\langle p\rangle = \frac{1}{i\hbar}\langle [p,H]\rangle \underset{I8.2}{=} -\left\langle \frac{d}{dx}V\right\rangle = -m\omega^2\langle x\rangle$

or directly with Ehrenfest

$\Rightarrow \frac{d^2}{dt^2}\langle x\rangle = -\omega^2\langle x\rangle$ same as for x_{cl}.

$\langle x\rangle(t) = A\cos\omega t + B\sin\omega t$; $\langle x\rangle(0) = \langle x\rangle_0 \Rightarrow A = \langle x\rangle_0$

$\langle p\rangle(0) = m\frac{d\langle x\rangle(0)}{dt} = \langle p\rangle_0 \Rightarrow \omega B = \frac{\langle p\rangle_0}{m}$

$\Rightarrow \langle x\rangle(t) = \langle x\rangle_0\cos\omega t + \frac{\langle p\rangle_0}{m\omega}\sin\omega t$

(b) $\frac{d}{dt}\langle T-V\rangle = \frac{1}{i\hbar}\langle T-V,H\rangle = \frac{1}{i\hbar}\langle [T,V]-[V,T]\rangle = \frac{2}{i\hbar}\left\langle \frac{p}{2m}\underbrace{[p,V]}_{-i\hbar\frac{dV}{dx}} + \underbrace{[p,V]}\frac{p}{2m}\right\rangle$

$= -\omega^2\langle px + xp\rangle$

$\frac{d}{dt}\langle px+xp\rangle = \frac{1}{i\hbar}\langle [px+xp,H]\rangle \underset{I.10}{=} 2\langle 2T - \underbrace{x\frac{dV}{dx}}_{2V}\rangle = 4\langle T-V\rangle$

$\Rightarrow \frac{d^2}{dt^2}\langle T-V\rangle = -4\omega^2\langle T-V\rangle$

$\Rightarrow \langle T-V\rangle(t) = A'\cos 2\omega t + B'\sin 2\omega t$; $\langle T-V\rangle(0) = A' =: \langle T-V\rangle_0$

$\frac{d}{dt}\langle T-V\rangle(0) = 2\omega B' = -\omega^2\langle px+xp\rangle_0$

$\Rightarrow \langle T-V\rangle(t) = (\langle T\rangle_0 - \langle V\rangle_0)\cos 2\omega t - \frac{\omega}{2}\langle px+xp\rangle_0 \sin 2\omega t$

Now $\langle x^2\rangle = \frac{2}{m\omega^2}\langle V\rangle = \frac{1}{m\omega^2}\langle H-(T-V)\rangle \underset{\langle H\rangle = E = \text{const.}}{=} \frac{1}{m\omega^2}\left(\langle T\rangle_0 + \langle V\rangle_0 - \langle T-V\rangle\right)$

$= \frac{1}{m\omega^2}\left(\langle T\rangle_0\underbrace{(1-\cos 2\omega t)}_{2\sin^2\omega t} + \langle V\rangle_0\underbrace{(1+\cos 2\omega t)}_{2\cos^2\omega t} + \frac{\omega}{2}\langle px+xp\rangle_0\sin 2\omega t\right)$

$= \frac{\langle p^2\rangle_0}{m^2\omega^2}\sin^2\omega t + \langle x^2\rangle_0\cos^2\omega t + \frac{1}{2m\omega}\langle px+xp\rangle_0\sin 2\omega t$

(c) $(\Delta x)^2(t) = \langle x^2\rangle_{(t)} - \langle x\rangle^2(t) = \frac{(\Delta p)_0^2}{m^2\omega^2}\sin^2\omega t + (\Delta x)_0^2\cos^2\omega t$

$+ \frac{1}{2m\omega}\left(\langle px+xp\rangle_0 - 2\langle x\rangle_0\langle p\rangle_0\right)\sin 2\omega t$

where

$(\Delta p)_0^2 = \langle p^2\rangle_0 - \langle p\rangle_0^2\ ;\ (\Delta x)_0^2 = \langle x^2\rangle_0 - \langle x\rangle_0^2$

Limit $\omega \to 0$: $\cos\omega t \to 1,\ \frac{1}{\omega}\sin\omega t \to t$

$(\Delta x)^2(t) \longrightarrow \frac{(\Delta p)_0^2}{m^2}t^2 + (\Delta x)_0^2 + \left(\langle px+xp\rangle_0 - 2\langle x\rangle_0\langle p\rangle_0\right)\frac{t}{m}$

as in HW IV, [3]

[2] (a) $\underbrace{\frac{\partial \rho_{cl}}{\partial t} = -\frac{d\vec{R}}{dt}}_{\text{chain rule}} \cdot \nabla\delta^{(3)}(\vec{r}-\vec{R}) = -\dot{\vec{R}}\cdot\nabla\delta^{(3)}(\vec{r}-\vec{R})$

$\nabla\vec{J}_{cl} = \dot{\vec{R}}\,\nabla\delta^{(3)}(\vec{r}-\vec{R})$

recall: $\vec{R}, \dot{\vec{R}}$ are indep. quantities to describe a system in cl. mechanics

$\Rightarrow \frac{\partial\rho_{cl}}{\partial t} + \nabla\vec{J}_{cl} = 0$

(b) $\langle\rho\rangle = \int\psi^*(\vec{r})\,\delta^{(3)}(\vec{r}-\vec{R})\,\psi(\vec{r})\,d^3r = \psi^*(\vec{R})\psi(\vec{R}) = \rho_{Schr.}(\vec{R})$ probability density of the Schrödinger field

$\langle\vec{J}\rangle = \frac{1}{2m}\int\psi^*(\vec{r})\left(-i\hbar\nabla_r\delta^{(3)}(\vec{r}-\vec{R}) + \delta^{(3)}(\vec{r}-\vec{R})(-i\hbar\nabla_r)\right)\psi(\vec{r})\,d^3r$

$= \frac{i\hbar}{2m}\left[\int(\nabla_r\psi)^*(\vec{r})\,\delta^{(3)}(\vec{r}-\vec{R})\,\psi(\vec{r})\,d^3r - \int\psi^*(\vec{r})\,\delta(\vec{r}-\vec{R})(\nabla_r\psi)(\vec{r})\,d^3r\right.$

$= \frac{i\hbar}{2m}\left[\nabla\psi^*(\vec{R})\,\psi(\vec{R}) - \psi^*(\vec{R})\,\nabla\psi(\vec{R})\right] = \vec{J}_{Schr.}(\vec{R})$

current for the Schrödinger field probability density

[3] (a) Hamilton–Jacobi: $\frac{\partial S}{\partial t} + \underbrace{\frac{1}{2m}\left(\nabla S - q\vec{A}\right)^2}_{H \text{ with } \vec{p} = \nabla S} + q\phi = 0$

Continuity eqn.: $\frac{\partial\rho}{\partial t} + (\nabla\vec{v})\,\rho + \vec{v}\cdot\nabla\rho = 0 \Rightarrow \frac{\partial\rho}{\partial t} + \frac{\rho}{m}\nabla(\vec{p}-q\vec{A}) + \frac{1}{m}\nabla\rho\cdot(\vec{p}-q\vec{A}) = 0$
(I.5.3)

(b) Ansatz $\psi(\vec r, t) = C e^{\frac{i}{\hbar}S}$ with C, S real into

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2m}\left(-i\hbar\nabla - q\vec A\right)^2 \psi + q\phi\psi$$

$$\Rightarrow \left(i\hbar \frac{1}{C}\frac{\partial C}{\partial t} - \frac{\partial S}{\partial t}\right)\psi = \frac{1}{2m}\Big(-\hbar^2\frac{\Delta C}{C} + (\nabla S)^2 - i\hbar\Delta S - i\hbar\frac{2}{C}\nabla C\cdot\nabla S + q^2A^2$$

$$+ i\hbar\frac{2}{C}q\vec A\cdot\nabla C - 2q\vec A\cdot\nabla S + i\hbar q\nabla\vec A\Big)\psi + q\phi\psi$$

Drop ψ from eq. and separate imaginary and real part:

$$\text{Re:}\quad 0 = \frac{\partial S}{\partial t} + \frac{1}{2m}\Big((\nabla S)^2 - \underbrace{\cancel{\hbar^2\frac{\Delta C}{C}}}_{0 \text{ for } \hbar\to 0} - 2\nabla S\cdot(q\vec A) + (q\vec A)^2\Big) + q\phi$$

$$\Rightarrow 0 = \frac{\partial S}{\partial t} + \frac{1}{2m}\left(\nabla S - q\vec A\right)^2 + q\phi \qquad \text{Hamilton-Jacobi!}$$

$$\text{Im:}\quad \underbrace{2C\frac{\partial C}{\partial t}}_{\frac{\partial\rho}{\partial t}} = \frac{1}{2m}\Big(-\underbrace{2C^2}_{\rho}\underbrace{\Delta S}_{\nabla\vec p} - \underbrace{4C\nabla C}_{2\nabla\rho}\cdot\underbrace{\nabla S}_{\vec p} + \underbrace{4C(\nabla C)}_{2\nabla\rho}\cdot(q\vec A) + \underbrace{2C^2}_{\rho}q\nabla\vec A\Big)$$

where $\rho = C^2 = |\psi|^2$; $\Rightarrow \frac{\partial\rho}{\partial t} + \frac{\rho}{m}\nabla(\vec p - q\vec A) + \frac{1}{m}\nabla\rho\cdot(\vec p - q\vec A) = 0$

continuity equation.

[4] Define $\vec A' = \vec A + \nabla f$; $\phi' = \phi - \frac{\partial f}{\partial t}$; $\psi' = \psi e^{\frac{i}{\hbar}qf}$

Proof of invariance in two parts:

i) $i\hbar\frac{\partial\psi'}{\partial t} - q\phi'\psi' = \left(i\hbar\frac{\partial\psi}{\partial t}\right)e^{\frac{i}{\hbar}qf} + \cancel{\left(-q\frac{\partial f}{\partial t}\right)\psi e^{\frac{i}{\hbar}qf}} - q\phi\psi e^{\frac{i}{\hbar}qf} + \cancel{q\frac{\partial f}{\partial t}\psi e^{\frac{i}{\hbar}qf}}$

$= \left(i\hbar\frac{\partial\psi}{\partial t} - q\phi\psi\right)e^{\frac{i}{\hbar}qf}$

ii) $\left(-i\hbar\nabla - q\vec A\right)^2\psi' = \left(-i\hbar\nabla - q\vec A\right)^2\psi e^{\frac{i}{\hbar}qf} + q^2(\nabla f)^2\psi e^{\frac{i}{\hbar}qf} + \left(-i\hbar\nabla - q\vec A\right)(-q\nabla f)\psi e^{\frac{i}{\hbar}qf}$

$+ (-q\nabla f)\left(-i\hbar\nabla - q\vec A\right)\psi e^{\frac{i}{\hbar}qf}$

$= \left[\left(-i\hbar\nabla - q\vec A\right)^2\psi\right]e^{\frac{i}{\hbar}qf} + 2\left[\left(-i\hbar\nabla - q\vec A\right)\psi\right]\overbrace{(-i\hbar\nabla)e^{\frac{i}{\hbar}qf}}^{q\nabla f e^{\frac{i}{\hbar}qf}} + \Big[\overbrace{(-i\hbar\nabla)^2 e^{\frac{i}{\hbar}qf}}^{q^2(\nabla f)^2 e^{\frac{i}{\hbar}qf} - i\hbar q\Delta f e^{\frac{i}{\hbar}qf}}\Big]\psi$

$+ \cancel{q^2(\nabla f)^2\psi e^{\frac{i}{\hbar}qf}} + \cancel{i\hbar q\Delta f\psi e^{\frac{i}{\hbar}qf}} - \cancel{2q^2(\nabla f)^2\psi e^{\frac{i}{\hbar}qf}} - \left[2q\nabla f\cdot\left(-i\hbar\nabla - q\vec A\right)\psi\right]e^{\frac{i}{\hbar}qf}$

④

$$= \left[\left(-i\hbar\nabla - q\vec{A}\right)^2 \psi \right] e^{\frac{i}{\hbar} q f}$$

$\Rightarrow$ the transformed S.E. $i\hbar \frac{\partial \psi'}{\partial t} = \frac{1}{2m}\left(-i\hbar\nabla - q\vec{A}'\right)^2 \psi' + q\phi'\psi'$

is identical to the original equation times an overall phase factor which can be dropped.

(b) $m \frac{d}{dt}\langle v_k \rangle = \frac{1}{i\hbar}\left\langle \left[p_k - qA_k, \frac{(\vec{P} - q\vec{A})^2}{2m} + q\phi \right] \right\rangle + m \left\langle \frac{\partial v_k}{\partial t} \right\rangle$ $\quad k = 1,2,3$

$$= \frac{1}{i\hbar}\left\langle (p_\ell - qA_\ell)[p_k - qA_k, p_\ell - qA_\ell]\frac{1}{2m} + \underbrace{[p_k - qA_k, p_\ell - qA_\ell]}_{i\hbar q\left(\frac{\partial}{\partial x_k}A_\ell - \frac{\partial}{\partial x_\ell}A_k\right)}(p_\ell - qA_\ell)\frac{1}{2m} \right\rangle$$

$$\underbrace{- q\left\langle \frac{\partial}{\partial x_k}\phi \right\rangle - q\left\langle \frac{\partial A_k}{\partial t} \right\rangle}_{= q\langle E_k \rangle}$$

$$= q\langle E_k \rangle + \frac{q}{2}\left\langle v_\ell\left(\frac{\partial}{\partial x_k}A_\ell - \frac{\partial}{\partial x_\ell}A_k\right) + \left(\frac{\partial}{\partial x_k}A_\ell - \frac{\partial}{\partial x_\ell}A_k\right)v_\ell \right\rangle$$

on the other hand $[\vec{v} \times (\nabla \times \vec{A})]_k = \varepsilon_{k\ell m} v_\ell \varepsilon_{mno} \frac{\partial}{\partial x^n} A_o$

$$= (\delta_{kn}\delta_{\ell o} - \delta_{ko}\delta_{\ell n}) v_\ell \frac{\partial}{\partial x^n} A_o = v_\ell\left(\frac{\partial}{\partial x_k}A_\ell - \frac{\partial}{\partial x_\ell}A_k\right)$$

$$\Rightarrow m\frac{d}{dt}\langle \vec{v} \rangle = q\langle \vec{E} \rangle + \frac{q}{2}\langle \vec{v} \times \vec{B} + \vec{B} \times \vec{v} \rangle$$