PHYS 606 — Spring 2014 — HW IV Solution

[1] $$[F_{\vec p}(\vec r), G_{\vec p}(\vec p)]\, f(\vec r) = (2\pi\hbar)^{-3/2} \int_{\mathbb{R}^3} [F(\vec r), G(-i\hbar\nabla_r)]\, \hat f(\vec p)\, e^{\frac{i}{\hbar}\vec r\cdot\vec p}\, d^3p$$

$$= (2\pi\hbar)^{-3/2} \int_{\mathbb{R}^3} \Big(F(\vec r)\, \hat f(\vec p)\, \underbrace{G(\vec p)\, e^{\frac{i}{\hbar}\vec r\cdot\vec p}}_{\text{power series of } G \text{ has acted on phase factor}} - G(-i\hbar\nabla_r)\, \hat f(\vec p)\, \underbrace{F(-i\hbar\nabla_p)\, e^{\frac{i}{\hbar}\vec r\cdot\vec p}}_{\text{power series of } F \text{ acting on phase factor}} \Big) d^3p$$

$$= (2\pi\hbar)^{-3/2} \int_{\mathbb{R}^3} \Big(G(\vec p)\, \hat f(\vec p)\, F(-i\hbar\nabla_p)\underline{e^{\frac{i}{\hbar}\vec r\cdot\vec p}} - \hat f(\vec p)\, F(-i\hbar\nabla_p)\, G(\vec p)\underline{e^{\frac{i}{\hbar}\vec r\cdot\vec p}} \Big) d^3p$$

$$= (2\pi\hbar)^{-3/2} \int_{\mathbb{R}^3} \Big(\underbrace{\big(F(+i\hbar\nabla_p)\, G(\vec p)\, \hat f(\vec p)\big)}_{\substack{\text{multiple partial integrations according to the power series}\\ \text{of } F\text{, boundary terms vanish for sufficiently fast falling functions.}}} e^{\frac{i}{\hbar}\vec r\cdot\vec p} - G(\vec p)\underbrace{\big(F(+i\hbar\nabla_p)\, \hat f(\vec p)\big)}_{\text{partial integrations}} e^{\frac{i}{\hbar}\vec r\cdot\vec p} \Big) d^3p$$

$$= (2\pi\hbar)^{-3/2} \int_{\mathbb{R}^3} [F_{\vec p}(\vec r), G_{\vec p}(\vec p)]\, \hat f(\vec p)\, e^{\frac{i}{\hbar}\vec r\cdot\vec p}\, d^3p$$

[2] (a) $$[\vec r\cdot\vec p, H] = p_i\left[\vec r\cdot\vec p, \frac{p_i}{2m}\right] + \left[\vec r\cdot\vec p, \frac{p_i}{2m}\right]p_i + r_i[p_i, V] + \overbrace{[r_i, V]}^{=0}\, p_i$$

$$= p_i r_j \underbrace{\left[p_j, \frac{p_i}{2m}\right]}_{=0} + p_i \underbrace{\left[r_j, \frac{p_i}{2m}\right]}_{\frac{2i\hbar}{2m}\delta_{ij}} p_j + \underbrace{\left[r_j, \frac{p_i}{2m}\right]}_{\frac{i\hbar}{2m}\delta_{ij}} p_j p_i + \vec r\cdot(-i\hbar\nabla V)$$

$$= i\hbar\frac{p^2}{m} - i\hbar\, \vec r\cdot\nabla V = i\hbar\,(2T - \vec r\,\nabla V)$$

$$[\vec p\cdot\vec r, H] = \text{... completely analogous ...} = i\hbar\,(2T - \vec r\cdot\nabla V)$$

(b) $$[(\vec r\cdot\vec p)(\vec r\cdot\vec p), T] = (\vec r\cdot\vec p)\underbrace{[\vec r\cdot\vec p, T]}_{(a)\ 2i\hbar T} + \underbrace{[\vec r\cdot\vec p, T]}_{(a)\ 2i\hbar T}\,\vec r\cdot\vec p = 2i\hbar\,(\vec r\cdot\vec p\, T + T\,\vec r\cdot\vec p)$$
$$= 2i\hbar\,\{\vec r\cdot\vec p, T\}$$

this sometimes denotes the anti commutator

$$[(\vec r\cdot\vec p)(\vec p\cdot\vec r), T] = 2i\hbar\,(\vec r\cdot\vec p\, T + T\,\vec p\cdot\vec r)$$

To see that both results are not the same apply these operators to test fcts. Here $\psi = e^{\frac{i}{\hbar}(\vec r\cdot\vec p - Et)}$ plane wave.

$$T\psi = -\hbar^2\frac{\Delta\psi}{2m} = \frac{p^2}{2m}\psi;\quad (\vec r\cdot\vec p)\psi = -i\hbar\,\vec r\cdot\nabla\psi = (\vec r\cdot\vec p)\psi$$

$\vec p$ operator!

$\vec p$ number, an eigenvalue of the operator $\vec p$!

$$(\vec p\cdot\vec r)\psi = \vec p\cdot\vec r\,\psi - i\hbar\underbrace{(\nabla\vec r)}_{3}\psi$$

operator — number

one $\vec p$ in T applied to $\vec r\cdot\vec p$

$$\Rightarrow [(\vec r\cdot\vec p)^2, T]\psi = 2i\hbar\left(\vec r\cdot\vec p\,\frac{p^2}{2m} + \frac{p^2}{2m}\,\vec r\cdot\vec p - 2i\hbar\frac{p^2}{2m}\right)\psi = \left(\frac{2i\hbar}{m}\vec r\cdot\vec p\, p^2 + \frac{2\hbar^2 p^2}{m}\right)\psi$$

all numbers

$$[(\vec r\cdot\vec p)(\vec p\cdot\vec r), T]\psi = 2i\hbar\left(\vec r\cdot\vec p\,\frac{p^2}{2m} + \frac{p^2}{2m}(\vec p\cdot\vec r - 3i\hbar)\right)\psi = \left(\frac{2i\hbar}{m}(\vec r\cdot\vec p)p^2 + \frac{5\hbar^2}{m}p^2\right)\psi$$

$-2i\hbar\frac{p^2}{2m}$

The two operators defined by the commutators are not the same.

(c) $$\{\vec r\cdot\vec p, H\} = \sum_{k=1}^{3}\Big(p_k\underbrace{\frac{\partial H}{\partial p_k}}_{p_k/m} - r_k\underbrace{\frac{\partial H}{\partial r_k}}_{\partial V/\partial r_k}\Big) = \frac{p^2}{m} - \vec r\cdot\nabla V = 2T - \vec r\cdot\nabla V$$

$$\{r_i, p_j\} = \sum_{k=1}^{3}(\delta_{ik}\delta_{jk} - 0) = \delta_{ij}$$

So it seems $[A, B] = i\hbar\,\{A, B\}$

operators — cl. quantities

(d) $$\{(\vec r\cdot\vec p)^2, T\} = \sum_{k=1}^{3}\left(2(\vec r\cdot\vec p)\,p_k\,\frac{p_k}{m} - 0\right) = 4\,\vec r\cdot\vec p\, T$$

Caveat: obviously the order matters: $\frac{1}{i\hbar}[(\vec r\cdot\vec p)^2, H] = 4\,\frac{1}{2}(\vec r\cdot\vec p\, T + T\,\vec r\cdot\vec p) \neq 4\,\vec r\cdot\vec p\, T$ for operators

[3] $\frac{d}{dt}\langle p\rangle = \underset{\text{(Ehrenfest)}}{\langle F\rangle} = 0$ (free particle) $\Rightarrow \langle p\rangle(t) = \underbrace{\langle p\rangle(0)}_{=:\, p_0 \text{ short notation}} = \text{const.}$

$$\frac{d}{dt}\langle x\rangle = \frac{\langle p\rangle}{m} = \frac{p_0}{m}$$

$$\Rightarrow \langle x\rangle(t) = \frac{p_0}{m}t + x_0 \qquad x_0 \equiv \langle x\rangle(0)$$

$$\frac{d}{dt}\langle p^2\rangle = \frac{1}{i\hbar}\langle[p,\underset{\frac{p^2}{2m}}{T}]\rangle = 0 \qquad \Rightarrow \langle p^2\rangle(t) = \langle p^2\rangle(0) = \text{const.}$$

$$\Rightarrow (\Delta p)^2(t) = \langle(p-\langle p\rangle)^2\rangle = \langle p^2\rangle(t) - (\langle p\rangle(t))^2 = \langle p^2\rangle(0) - p_0^2 = (\Delta p)^2(0) = \text{const.}$$

$$\frac{d}{dt}\langle x^2\rangle = \frac{1}{i\hbar}\langle[x^2,\tfrac{p^2}{2m}]\rangle = \frac{1}{i\hbar}\langle p[x^2,\tfrac{p}{2m}] + [x^2,\tfrac{p}{2m}]p\rangle$$

$$= \frac{1}{i\hbar}\langle px[x,p] + p[x,p]x + x[x,p]p + [x,p]xp\rangle\frac{1}{2m}$$

$$= \frac{1}{m}\langle px + xp\rangle$$

$$\frac{d}{dt}\langle px\rangle = \frac{1}{i\hbar}\langle[px,T]\rangle \overset{[2](a)}{=} 2T = \text{const.} = \frac{d}{dt}\langle xp\rangle$$

$$\Rightarrow \langle px+xp\rangle(t) = \frac{2\langle p^2\rangle(0)}{m}t + \langle px+xp\rangle(0)$$

$$\Rightarrow \langle x^2\rangle = \frac{\langle p^2\rangle(0)}{m^2}t^2 + \frac{1}{m}\langle px+xp\rangle(0)\,t + \langle x^2\rangle(0)$$

$$\Rightarrow (\Delta x)^2(t) = \langle x^2\rangle(t) - (\langle x\rangle(t))^2 = \frac{\langle p^2\rangle(0) - p_0^2}{m^2}t^2 + \frac{t}{m}\left(\langle px+xp\rangle(0) - 2x_0p_0\right)$$
$$+ \langle x^2\rangle(0) - x_0^2$$

$$= \frac{(\Delta p)^2(t)}{m^2}t^2 + \frac{2}{m}\left(\frac{1}{2}\langle px+xp\rangle(0) - \langle p\rangle(0)\langle x\rangle(0)\right)t + (\Delta x)^2(0)$$

[4] (a) Gauss: $(\Delta x)^2(0) = \sigma^2$, $(\Delta p)^2(0) = \frac{\hbar^2}{4\sigma^2}$ (HW I, [3])

$$\langle x\rangle(0) = x_0, \quad \langle p\rangle(0) = \hbar k_0 \equiv p_0$$

$$\langle xp\rangle(0) = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{\mathbb{R}} e^{-\frac{(x-x_0)^2}{4\sigma^2}} e^{-\frac{i}{\hbar}p_0x}\, x\left(-i\hbar\frac{d}{dx}\right) e^{-\frac{(x-x_0)^2}{4\sigma^2}} e^{+\frac{i}{\hbar}p_0x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma}\int_{\mathbb{R}} e^{-\frac{(x-x_0)^2}{2\sigma^2}}\, x\left(p_0 + i\hbar\frac{x-x_0}{2\sigma^2}\right)dx$$

$$\underset{u=x-x_0}{=} \frac{1}{\sqrt{2\pi}\sigma}\int e^{-\frac{u^2}{2\sigma^2}}\left(up_0 + i\hbar\frac{u^2}{2\sigma^2} + x_0 p_0 + i\hbar x_0\frac{u}{2\sigma^2}\right) d$$

$$= x_0 p_0 + i\hbar\frac{\sigma^2}{2\sigma^2} = x_0 p_0 + \frac{1}{2} i\hbar$$

$\langle px\rangle(0) =$ same as above + term with $(-i\hbar\frac{d}{dx})$ acting on x

$$= x_0 p_0 + \frac{1}{2}i\hbar + \frac{1}{\sqrt{2\pi}\sigma}\int_{\mathbb{R}} e^{-\frac{u^2}{2\sigma^2}}(-i\hbar) = x_0 p_0 - \frac{1}{2}i\hbar$$

$$\Rightarrow \frac{1}{2}\langle px + xp\rangle(0) = x_0 p_0$$

$$\Rightarrow (\Delta x)^2(t) = \frac{\hbar^2}{4\sigma^2 m^2}t^2 + \sigma^2 \quad \text{and} \quad (\Delta p)^2(t) = \frac{\hbar^2}{4\sigma^2}$$

(b) $$(\Delta p)^2(t) \overset{HW\,II,\,[1](a)}{=} \frac{1}{\sqrt{2\pi}\hat\sigma}\int e^{-\frac{(p-p_0)^2}{4\hat\sigma^2}}\left|e^{-\frac{i}{\hbar}(p-p_0)x_0}e^{-i\omega(p)t}\right|^2(p-p_0)^2\,dp \quad \left(\hat\sigma = \frac{\hbar}{2\sigma}\right)$$

$$= (\Delta p)^2(0) = \hat\sigma^2 = \frac{\hbar^2}{4\sigma^2}$$

HW II, [1](a): $$|\psi(x,t)|^2 = \frac{1}{\sqrt{2\pi}}\frac{\sigma}{\sqrt{\sigma^4 + \frac{\hbar^2 t^2}{4m^2}}}\, e^{-\frac{(x-x_0-v_{gr}t)^2\sigma^2}{2(\sigma^4+\frac{\hbar^2t^2}{4m^2})}} \qquad v_{gr} = \frac{p_0}{m}$$

$$\langle x\rangle(t) \underset{x-x_0-v_{gr}t=:u}{=} \frac{1}{\sqrt{2\pi}}\frac{\sigma}{\sqrt{\sigma^4+\frac{\hbar^2t^2}{4m^2}}}\int_{\mathbb{R}} du\,\underbrace{(x_0 + v_{gr}t + u)}_{=x}\, e^{-\frac{u^2\sigma^2}{2(\sigma^4+\frac{\hbar^2t^2}{4m^2})}} = x_0 + v_{gr}t$$

$$\langle x^2\rangle(t) = \frac{1}{\sqrt{2\pi}}\frac{\sigma}{\sqrt{\sigma^4+\frac{\hbar^2t^2}{4m^2}}}\int_{\mathbb{R}} du\left((x_0+v_{gr}t)^2 + 2u(x_0+v_{gr}t) + u^2\right)e^{-\frac{u^2\sigma^2}{2(\sigma^4+\frac{\hbar^2t^2}{4m^2})}}$$

$$= (x_0 + v_{gr}t)^2 + \frac{\sigma^4 + \frac{\hbar^2t^2}{4m^2}}{\sigma^2}$$

$$\Rightarrow \langle(x-\langle x\rangle)^2\rangle = \langle x^2\rangle(t) - (\langle x\rangle(t))^2 = \sigma^2 + \frac{\hbar^2}{4\sigma^2m^2}t^2$$