PHYS 606 – Spring 2014 – HW III Solution

[1] We have $i\hbar \frac{\partial}{\partial t}\psi(\vec{r},t) = -\frac{\hbar^2}{2m}\Delta\psi(\vec{r},t) + V(\vec{r})\,\psi(\vec{r},t)$

Using the Fourier transformation $\psi(\vec{r},t) = \frac{1}{(2\pi\hbar)^{3/2}}\int_{\mathbb{R}^3} \phi(\vec{p},t)\, e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}\, d^3p$

we get

$$\frac{1}{(2\pi\hbar)^{3/2}}\int_{\mathbb{R}^3}\left[i\hbar\frac{\partial}{\partial t}\phi(\vec{p},t)\right]e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}} = \frac{1}{(2\pi\hbar)^{3/2}}\int_{\mathbb{R}^3}\phi(\vec{p},t)\Big[\underbrace{\left(-\frac{\hbar^2}{2m}\Delta\right)e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}}_{=\frac{p^2}{2m}e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}} + \underbrace{V(\vec{r})\,e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}}_{V(-\frac{i}{\hbar}\nabla_p)e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}}\Big]d^3p$$

from a power series $r_k^n e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}} = \left(-\frac{i}{\hbar}\frac{\partial}{\partial p_k}\right)^n e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}$, $k=1,2,3$; $n\in\mathbb{N}$

partial integration in the last term: $\int \phi\, \nabla_p\, e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}\, d^3p =$

$= \int(-\nabla_p\phi)\,e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}\,d^3p\,+$ boundary term $(\to 0)$

after repeated application to the power series expansion of V:

$$\frac{1}{(2\pi\hbar)^{3/2}}\int_{\mathbb{R}^3}\left[i\hbar\frac{\partial}{\partial t}\phi(\vec{p},t) - \frac{p^2}{2m}\phi(\vec{p},t) - V\!\left(\frac{i}{\hbar}\nabla_p\right)\phi(\vec{p},t)\right]e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}\,d^3p = 0$$

$\Rightarrow$ Expression in bracket $[\dots] = 0$ (apply inverse FT!)

[2] (a) Induction: eqs. (2)+(3) obviously true for $n=1$; suppose they are true for $n-1$

then
$$\begin{cases}[F,G^n] = G^{n-1}[F,G] + [F,G^{n-1}]G = G^{n-1}[F,G] + (n-2)G^{n-2}[F,G]G \\ [F^n,G] \underset{\uparrow}{=} F^{n-1}[F,G] + [F^{n-1},G]F = F^{n-1}[F,G] + (n-2)F^{n-2}[F,G]F\end{cases}$$

G commutes w/ [F,G]

Product rule

$$= \begin{cases}(n-1)\,G^{n-1}[F,G] \\ (n-1)\,F^{n-1}[F,G]\end{cases}$$

Direct calculation is also possible.

(b) $$[F,[G,H]] = [F,GH] - [F,HG] \underset{\text{Product rule}}{=} G[F,H] + [F,G]H - H[F,G] - [F,H]G$$

$$= [G,[F,H]] - [H,[F,G]] = -[H,[F,G]] - [G,[H,F]]$$

[3] (a) In $\vec p$-space: $[\vec L, T] = \left[i\hbar \nabla_p \times \vec p, \frac{p^2}{2m}\right]$

x-component: $$\left[i\hbar\left(\frac{\partial}{\partial p_y} p_z - \frac{\partial}{\partial p_z} p_y\right), \frac{\sum_{i=1}^{3} p_i^2}{2m}\right] = \frac{i\hbar}{2m}\Bigg(\frac{\partial}{\partial p_y}\underbrace{\Big[p_z,\sum_i p_i^2\Big]}_{=0} + \underbrace{\Big[\frac{\partial}{\partial p_y},\sum_i p_i^2\Big]}_{2p_y} p_z$$
$$- \frac{\partial}{\partial p_z}\underbrace{\Big[p_y,\sum_i p_i^2\Big]}_{=0} - \underbrace{\Big[\frac{\partial}{\partial p_z},\sum_i p_i^2\Big]}_{2p_z} p_y\Bigg) = 0$$

Similar for y- and z-components.

In $\vec r$-space: $[\vec L, T] = \left[\vec r \times (-i\hbar\nabla_r), -\frac{\hbar^2}{2m}\Delta\right]$

x-component: $$\left[-i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right), -\frac{\hbar^2}{2m}\Delta\right]$$
$$= \frac{i\hbar^3}{2m}\Bigg(y\underbrace{\Big[\frac{\partial}{\partial z},\Delta\Big]}_{=0} + \underbrace{[y,\Delta]}_{2\frac{\partial}{\partial y}}\frac{\partial}{\partial z} - z\underbrace{\Big[\frac{\partial}{\partial y},\Delta\Big]}_{=0} - \underbrace{[z,\Delta]}_{2\frac{\partial}{\partial z}}\frac{\partial}{\partial y}\Bigg) = 0$$

(b) $$\frac{d}{dt}\langle \vec L\rangle = \int_{\mathbb{R}^3} \frac{\partial\psi^*}{\partial t}\vec L\psi\, d^3r + \int_{\mathbb{R}^3}\psi^*\underbrace{\frac{\partial \vec L}{\partial t}}_{=0}\psi\, d^3r + \int_{\mathbb{R}^3}\psi^*\vec L\frac{\partial\psi}{\partial t}\, d^3r$$

$$\underset{\text{Schr. equ.}}{=} \int_{\mathbb{R}^3}\underbrace{\frac{1}{-i\hbar}(H\psi^*)\vec L\psi}_{\text{partial integr.}}\, d^3r + \frac{1}{i\hbar}\int_{\mathbb{R}^3}\psi^*\vec L H\psi\, d^3r = \int_{\mathbb{R}^3}\frac{1}{i\hbar}\psi^*[\vec L,H]\psi\, d^3r$$

$\int \Delta\psi^*\vec L\psi = \int\psi^*\Delta\vec L\psi$ + boundary terms $\to 0$

$$\underset{(a)}{=} \frac{1}{i\hbar}\langle[\vec L,V]\rangle = -\langle[\vec r\times\nabla, V]\rangle = \langle \vec r\times\vec F\rangle$$

[4] (a) Merzbacher p. 39 has an elegant proof. Here let's try brute force math.

After looking at the first couple of nested commutators of type $[F,G]$, $[F,[F,G]]$, $[F[F,[F,G]]]$ etc. the following formula suggests itself:

$$\underbrace{[F,[F,\dots,[F,G]]\dots]}_{k \text{ commutators with } F} = \sum_{j=0}^{k} (-1)^j \binom{k}{j} F^{k-j} G F^j$$

Proof by induction: $k=1$ clear; if formula holds for $k-1$ we have for k:

$$F\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}F^{k-1-j}GF^j - \sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}F^{k-1-j}GF^jF$$

$j=0$ term, $j\neq 0$ terms, $j-1$ terms, $j=k-1$ term

$$= F^kG + \sum_{j=1}^{k-1} F^{k-j}GF^j(-1)^j\underbrace{\left[\binom{k-1}{j}+\binom{k-1}{j}\right]}_{\binom{k}{j}} - GF^k(-1)^{k-1}$$

$\square$

On the other hand

$$e^FGe^{-F} = \left(\sum_{k=0}^{\infty} \frac{1}{k!}F^k\right)G\left(\sum_{k'=0}^{\infty}(-1)^{k'}\frac{1}{k'!}F^{k'}\right)$$

$$= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{j=0}^{k} F^{k-j}\frac{1}{(k-j)!}GF^j(-1)^j\frac{1}{j!}k!\right)$$

$$= \sum_{k=0}^{\infty}\frac{1}{k!}\underbrace{[F,\dots,[F,G]]\dots]}_{k \text{ times}}$$

$\square$

(b) Let's use the trick suggested by Merzbacher (other ways, e.g. calculating $\log(e^Fe^G)$ using the series for the logarithm can be found in the literature.)

For $t\in\mathbb{R}$ Consider

$$\frac{d}{dt}e^{tF}e^{tG} = e^{tF}Fe^{tG} + \underbrace{e^{tF}G}e^{tG} = \left(F+G+\frac{1}{2}[F,G]\right)e^{tF}e^{tG}$$

usual product rule or explicitly via power series

$= \left(G+\frac{1}{2!}[F,G]\right)e^{tF}$ According to (a)! Higher order commutators vanish

This is a diff. equation $\frac{d\varphi}{dt} = \text{const} * \varphi$ with $\varphi = e^{tF}e^{tG}$. The solution must be of the form $e^{t*\text{const}}$.

$$\Rightarrow e^{tF}e^{tG} = e^{t(F+G+\frac{1}{2}[F,G])}.$$ Now set $t=1$.