PHYS 606 — Spring 2014 — HW I Solution

[1] (a) Hamilton fct. $H = \frac{p^2}{2m} + \frac{k}{2}x^2 = E \Rightarrow \frac{p^2}{2mE} + \frac{x^2}{\frac{2E}{k}} = 1$

$\Rightarrow$ Phase space motion is ellipse with semi axes $\sqrt{2mE}$ and $\sqrt{\frac{2E}{k}}$

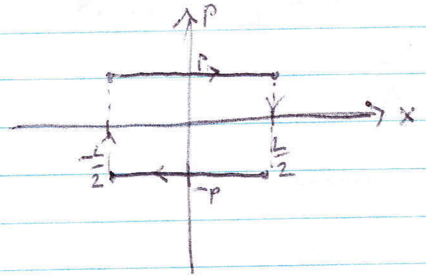
Bohr–Sommerfeld: $\underbrace{\oint p\,dq = \pi\sqrt{2mE}\sqrt{\frac{2E}{k}}}_{\text{area of ellipse}} = 2\pi\frac{E}{\omega}$ with $\omega = \sqrt{\frac{k}{m}}$

On the other hand $\oint p\,dq \doteq nh$

$\Rightarrow E = n\frac{h\omega}{2\pi} = n\hbar\omega$

All these energy levels are off by $\frac{1}{2}\hbar\omega$ from the full QM result, but the radiation spectrum (involving ΔE) can be predicted accurately.

(b) Consider particle with momentum p (moving right); reflected at $x = +\frac{L}{2}$ to obtain momentum $-p$ (energy conserved); another reflection $-p \to p$ at $x = -\frac{L}{2}$

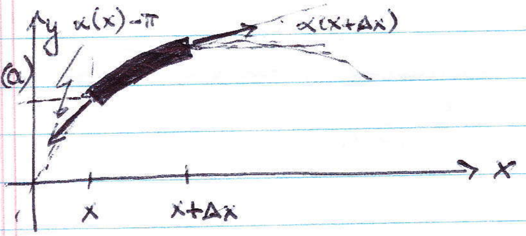


$\oint p\,dq = 2pL \doteq nh$

$\Rightarrow E = \frac{p^2}{2m} = n^2\frac{h^2}{8mL^2}$

same as the full QM result!

[2] (a)



Net force on small segment:

$$\Delta\vec{F} = T\left[\sin\alpha(x+\Delta x) - \sin\alpha(x)\right]\hat{j}$$
$$+ T\left[\cos\alpha(x+\Delta x) - \cos\alpha(x)\right]\hat{i}$$

Small displacements ~ small α's $\Rightarrow \begin{cases} \sin\alpha \approx 1 \approx \tan\alpha = \frac{\partial y}{\partial x} \\ \cos\alpha \approx 1 \quad \text{(i.e. no longitud displacements)} \end{cases}$

$$\Rightarrow \Delta F_y \approx T\left[\frac{\partial y}{\partial x}(x+\Delta x) - \frac{\partial y}{\partial x}(x)\right] \xrightarrow[\Delta x \to 0]{} dF_y = T\frac{\partial^2 y}{\partial x^2}\,dx$$

On the other hand $dF_y = dm\,\ddot{y} = \rho\ddot{y}\,dx$ (Newton's 2^{nd} Law)

$$\Rightarrow \frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho} \frac{\partial^2 y}{\partial x^2}$$

classical wave equation with wave speed $v = \sqrt{\frac{T}{\rho}}$

(b) Use $y(x,t) = W(x)\, h(t) \Longrightarrow \frac{\ddot{h}(t)}{h(t)} = v^2 \frac{W''(x)}{W(x)}$

Separation of variables! $\Rightarrow \begin{cases} \ddot{h} = -K v^2 h \\ W'' = -K W \end{cases}$

Two harmonic oscillators: $W = A \sin(\sqrt{K} x + \delta)$

$h = \cos(\sqrt{K} v t + \varepsilon)$

this normalization factor can be absorbed in A!

(here use $K > 0$, for $K < 0$ solutions are exponentials which could not fulfill the boundary conditions)

Boundary conditions, e.g. $\left.\begin{array}{l} W(0) = 0 \\ W(L) = 0 \end{array}\right\} \Rightarrow \delta = 0$ (or multiples of π which can be dropped)

and $\sqrt{K} = \frac{n\pi}{L} \quad n \in \mathbb{N}$

$\Rightarrow$ Allowed wave lengths $\lambda = \frac{2\pi}{\frac{n\pi}{L}} = \frac{2L}{n}$

$\Rightarrow$ General solution for these boundary conditions:

$$y(x,t) = \sum_{n \in \mathbb{N}} A_n \sin\left(\frac{n\pi}{L} x\right) \cos\left(\frac{n\pi}{L} v t + \varepsilon_n\right)$$

Coefficients A_n and phases ε_n could be determined from the Fourier series of a given initial condition $y(x,0)$, $\dot{y}(x,0)$.

(c) $k = \frac{n\pi}{L} \Rightarrow E = \frac{p^2}{2m} = \frac{\hbar^2}{2m} \frac{n^2 \pi^2}{L^2} = n^2 \frac{h^2}{8mL^2}$

same as in the QM treatment of the potential well.

[3] (a) $$|C|^2 \int_{\mathbb{R}} e^{-\frac{(x-x_0)^2}{2\sigma^2}}\,dx \underset{z=\frac{x-x_0}{\sqrt{2}\sigma}}{=} |C|^2 \int_{\mathbb{R}} e^{-z^2}dz \cdot \sqrt{2}\sigma = |C|^2\sqrt{2\pi}\,\sigma$$

[One bonus point if you calculated that integral yourself:

$$\int_{\mathbb{R}} e^{-z^2}dz = \left(\int_{\mathbb{R}^2} e^{-r^2}d^2r\right)^{1/2} = \left(2\pi\int_0^\infty r e^{-r^2}dr\right)^{1/2} \underset{u=r^2}{=} \left(\pi\int_0^\infty e^{-u}du\right)^{1/2} = \sqrt{\pi}$$]

$$\Rightarrow C = \frac{1}{\sqrt{\sqrt{2\pi}\,\sigma}}$$ ($\times$ a phase which we ignore here)

(b) $$\langle x\rangle = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{\mathbb{R}} x\, e^{-\frac{(x-x_0)^2}{2\sigma^2}}dx \underset{u=\frac{x-x_0}{\sqrt{2}\sigma}}{=} \frac{1}{\sqrt{2\pi}\,\sigma}\int_{\mathbb{R}}(\sqrt{2}\sigma u + x_0)e^{-u^2}\sqrt{2}\sigma\,du$$

$$\underset{\int u e^{-u^2}du=0}{=} \frac{x_0}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-u^2}du = x_0$$

$$(\Delta x)^2 = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{\mathbb{R}}(x-x_0)^2 e^{-\frac{(x-x_0)^2}{2\sigma^2}}dx \underset{u=\frac{x-x_0}{\sqrt{2}\sigma}}{=} \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} 2\sigma^2u^2e^{-u^2}du = -\frac{\sigma^2}{\sqrt{\pi}}\int_{\mathbb{R}} u\frac{d}{du}e^{-u^2}du$$

$$\underset{\text{part. int}}{=} \Big[\underbrace{-\frac{\sigma^2}{\sqrt{\pi}}ue^{-u^2}}_{0}\Big]_{-\infty}^{+\infty} + \frac{\sigma^2}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-u^2}du = \sigma^2$$

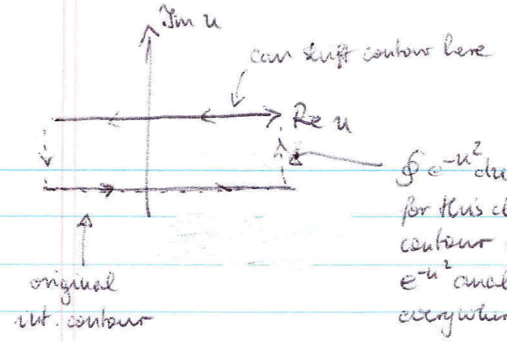
Center x_0 and width σ are equal to "average x" and $\sqrt{\text{variance}} = \Delta x$, resp.

(c) $$\hat f(k) = (2\pi)^{-1/2}\frac{1}{\sqrt{\sqrt{2\pi}\,\sigma}}\int_{\mathbb{R}} e^{-\frac{(x-x_0)^2}{4\sigma^2}}e^{-ikx}\,dx$$

$$= \frac{1}{(2\pi)^{3/4}\sigma^{1/2}}e^{-ikx_0}\int_{\mathbb{R}} e^{-\frac{x^2}{4\sigma^2}}e^{-ikx}dx$$

Complete square → $$= \frac{1}{(2\pi)^{3/4}\sigma^{1/2}}e^{-ikx_0}\int_{\mathbb{R}} e^{-\left(\frac{x}{2\sigma}+ik\sigma\right)^2}dx\; e^{-\sigma^2k^2}$$

$$\underset{u=\frac{x}{2\sigma}+ik\sigma}{=} \frac{2^{1/4}\sigma^{1/2}}{\pi^{3/4}}e^{-ikx_0}e^{-\sigma^2k^2}\int_{-\infty-2ik\sigma^2}^{+\infty-2ik\sigma^2} e^{-u^2}du = \sqrt[4]{\frac{2}{\pi}}\sqrt{\sigma}\,e^{-ikx_0}e^{-\sigma^2k^2}$$



$\oint e^{-u^2} du = 0$ for this closed contour since e^{-u^2} analytic everywhere

so $\displaystyle\int_{-\infty-2ik\sigma^2}^{+\infty-2ik\sigma^2} e^{-u^2} du = \int_{-\infty}^{+\infty} e^{-u^2} du = \sqrt{\pi}$

We can write this in "standard form" with a width in momentum space

$$\hat{\sigma} = \frac{1}{2\sigma} \quad ; \text{ then } \quad \hat{f}(k) = \frac{1}{\sqrt{\sqrt{2\pi}\,\hat{\sigma}}}\, e^{-ikx_0}\, e^{-\frac{k^2}{4\hat{\sigma}^2}}$$

[Summary: Gauss centered around 0, width σ $\xleftrightarrow{FT}$ Gauss centered around 0, width $\frac{1}{2\sigma}$

Displacement by x_0 $\xleftrightarrow{FT}$ phase factor e^{-ikx_0}]

Using the result from (b): $\langle k \rangle = 0$

(phase e^{-ikx_0} drops out) $\quad (\Delta k)^2 = \langle k^2 \rangle = \sigma_k^2 = \frac{1}{4\sigma^2}$

$$\Rightarrow \Delta x\, \Delta k = \sigma\, \sigma_k = \frac{1}{2}$$

[4] (a) $\displaystyle\int_{\mathbb{R}} |\phi(k)|^2 dk = |C|^2 a \implies C = \frac{1}{\sqrt{a}}$ modulo a phase

other sign also accepted here; problem text should have read "inverse FT"

$$\psi(x) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \phi(k)\, e^{ikx}\, dk = (2\pi a)^{-1/2}\, e^{ik_0 x} \int_{-a/2}^{a/2} e^{iux}\, du = \sqrt{\frac{2}{\pi a}}\, \frac{\sin \frac{a}{2} x}{x}$$

"sinc wave packet"

Plots: see below

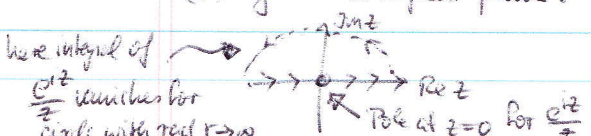
(b) $\displaystyle\int_{\mathbb{R}} |\psi(x)|^2 dx \underset{u=\frac{a}{2}x}{=} \frac{1}{\pi}\int_{\mathbb{R}} \frac{\sin^2 u}{u^2}\, du = 1$ and $\displaystyle\int_{\mathbb{R}} |\phi(k)| d k = 1 \Rightarrow$ Plancherel!

5 Bonus points for solving the sinc²-integral explicitly; (i) reduce sinc² to sinc

$$\int_{\mathbb{R}} \frac{\sin^2 u}{u^2}\, du = \int_{\mathbb{R}} \sin^2 u \,\frac{d}{du}\left(-\frac{1}{u}\right) du \underset{P.I.}{=} \left[-\frac{\sin^2 u}{u}\right]_{-\infty}^{+\infty} + \int_{\mathbb{R}} \frac{\sin 2u}{u}\, du = \int_{\mathbb{R}} \frac{\sin u}{u}\, du \; (!)$$

(ii) go to complex plane: $\displaystyle\int_{\mathbb{R}} \frac{\sin u}{u}\, du = \mathrm{Im} \int_{\mathbb{R}} \frac{e^{iz}}{z}\, dz = \mathrm{Im} \oint \frac{e^{iz}}{z}\, dz$

integral closed in upper half plane



$$= \operatorname{Im}\left(\frac{i}{2}\,2\pi i \operatorname*{Res}_{z=0} \frac{e^{iz}}{z}\right) = \operatorname{Im}(\pi i) = \pi$$

$\frac{1}{2}$ of residue at pole $z=0$ contributes

$\lim_{z\to 0} \frac{e^{iz}}{z} = \frac{1}{z}$

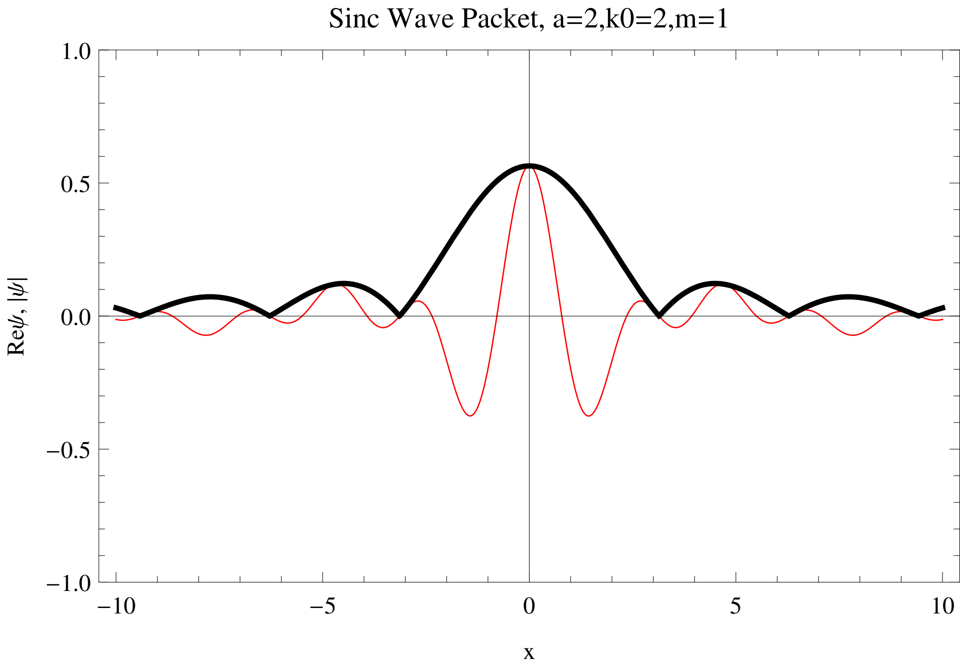
(c) $$(\Delta k)^2 = \int_{\mathbb{R}} (k-k_0)^2 |\phi(k)|^2 dk = \frac{1}{a}\int_{-\frac{a}{2}}^{+\frac{a}{2}} k^2 dk = \frac{1}{12}a^2$$

$$(\Delta x)^2 = \int_{\mathbb{R}} (x-x_0)^2 |\psi(x)|^2 dx \underset{x_0=0}{=} \frac{2}{\pi a}\int_{\mathbb{R}} \sin^2 \frac{a}{2}x \, dx \qquad \text{divergent!}$$

The width of the sinc wave packet can not be defined through its variance.

One possible measure of its width: the first zero $\frac{a}{2}x = \pi \Rightarrow \Delta x \sim \frac{2\pi}{a}$

$\Rightarrow (\Delta x)(\Delta k) \sim \frac{\pi}{\sqrt{3}} \sim O(1)$ independent of a!


Sinc Wave Packet, a=2,k0=2,m=1
1.0
0.5
0.0
−0.5
−1.0
Reψ, |ψ|
−10
−5
0
5
10
x