Physics 606 (Quantum Mechanics I) — Spring 2014

Midterm Exam

Instructor: Rainer J. Fries

[1] **Complex Potential** (30 points)

Sometimes it is useful to allow the potential energy in the time-dependent Schrödinger equation to be complex, i.e. $V(\vec{r}) = V'(\vec{r}) - iV''(\vec{r})$ where both V' and V'' are real.

(a) (15) Using the usual ansatz $\psi(\vec{r}, t) = A(\vec{r}, t) e^{\frac{i}{\hbar} S(\vec{r}, t)}$ in the Schrödinger equation with real-valued amplitude A and phase S *derive* the modified Hamilton-Jacobi equation and continuity equation for S and the particle density $\rho = A^2$ in the limit $\hbar \to 0$ in this case.

(b) (5) Discuss the differences compared to the known case of a purely real potential ($V'' = 0$). How can the additional terms involving V'' be interpreted?

(c) (10) Consider the case of vanishing real potential $V' = 0$ and *constant* and small imaginary potential V''. How are the plane wave solutions of the free time-dependent Schrödinger equation modified by the presence of this small imaginary potential? "Small" here means that $|V''|$ is small compared to the energy $\hbar\omega$ of the plane waves considered.
Hint: Neglect the $\vec{r}$-dependence of A in the ansatz given in (a).

[2] **Half Oscillator** (25 points)

Consider a particle of mass m with potential energy

$$V(x) = \begin{cases} \infty & \text{for } x < 0 \\ \frac{1}{2} m\omega^2 x^2 & \text{for } x > 0 \end{cases} \tag{1}$$

i.e. a "halved" harmonic oscillator. Find the energy eigenvalues and properly normalized eigenfunctions for this particle.
Hint: Make good use of the results of the "full" harmonic oscillator.

[3] **Ehrenfest Theorem: Newton's Law and Quantum Corrections** (20 points)

Consider a particle of mass m with potential energy $V(\vec{r})$. Ehrenfest's Theorem is an exact statement about the motion of the average position $\langle \vec{r} \rangle$. Using the assumption $\langle V(\vec{r}) \rangle \approx V(\langle \vec{r} \rangle)$ one famously recovers Newton's Second Law for the average position.

(a) (5) Rederive the exact version of Ehrenfest's Theorem from the general equation of motion for expectation values.

(b) (15) Now assume that the potential energy $V(\vec{r})$ is *slowly varying* as a function of position $\vec{r}$. Using Ehrenfest's Theorem from (a) derive Newton's Second Law for $\langle \vec{r} \rangle$ and the first quantum correction to it.

[4] **Skewness** (25 points)

The skewness (third moment) of a distribution in a variable x is usually defined as

$$\gamma_1^x = \frac{1}{(\Delta x)^3} \left\langle (x - \langle x \rangle)^3 \right\rangle \ . \tag{2}$$

Consider a free particle of mass m described by a wave packet.

(a) (10) Calculate the skewness $\gamma_1^p(t)$ for the distribution of *momenta* p in the wave packet (definition of skewness of p is analagous to the skewness of x) as a function of time t, given its initial value $\gamma_1^p(0)$ at time $t = 0$.

(b) (15) Derive a differential equation for $\langle x^3 \rangle$ as a function of time t by differentiating sufficiently often.
Note: You could integrate this result from given initial conditions to obtain $\gamma_1^x(t)$ but you don't have to do that here.

Useful Formulae

- δ-function

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{ik(x-x_0)}dk = \delta(x-x_0) \tag{3}$$

- Hamilton-Jacobi for the classical action $S(\vec{r},\vec{p},t)$

$$\frac{\partial S}{\partial t} + H(\vec{r},\vec{p}) = 0 \quad \text{with}\, p_i = \frac{\partial S}{\partial r_i} \tag{4}$$

- Current of the Schrödinger field

$$\vec{j}(\vec{r},t) = \frac{\hbar}{2mi}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) \tag{5}$$

- Jacobi identity

$$[F,[G,H]] + [H,[F,G]] + [G,[H,F]] = 0 \tag{6}$$

- Baker Campbell Hausdorff (if A, B commute with their commutator)

$$e^A e^B = e^{A+B+[A,B]/2} \tag{7}$$

- Virial theorem for stationary states

$$2\langle T\rangle = \langle \vec{r}\cdot\nabla V\rangle \tag{8}$$

- Closure/completeness for continuous spectrum with eigenstates ψ_α

$$\int_{\text{spec}} \psi_\alpha^*(\vec{r})\psi_\alpha(\vec{r})d\alpha \tag{9}$$

- Generator of Galilei boosts

$$\vec{K} = m\vec{r} - \vec{p}t \tag{10}$$

- Hermite polynomials

$$\frac{d^2}{d\xi^2}H_n(\xi) - 2\xi\frac{d}{d\xi}H_n(\xi) + 2nH_n(\xi) \tag{11}$$

$$\frac{d}{d\xi}H_n(\xi) = 2nH_{n-1}(\xi) \tag{12}$$

$$F(\xi,s) = \sum_{n\in\mathbb{N}} H_n(\xi)\frac{s^n}{n!} = e^{\xi^2-(s-\xi)^2} \tag{13}$$

- Harmonic oscillator: orthonormal energy eigenstates

$$\psi_n(x) = 2^{-\frac{n}{2}} n!^{-\frac{1}{2}} \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) e^{-\frac{m\omega}{2\hbar}x^2} \tag{14}$$