Physics 606 (Quantum Mechanics I) — Spring 2014

Homework 9

Instructor: Rainer J. Fries *Turn in your work by April 10*

[1] **Legendre Polynomials and Legendre Functions** (40 points)

Consider the differential equation

$$\frac{d}{d\xi}\left((1-\xi^2)\frac{dP}{d\xi}\right) - \frac{m^2}{1-\xi^2}P + \lambda P = 0 \tag{1}$$

for a function $P(\xi)$, $-1 < \xi < 1$, with parameters $m \in \mathbb{N}$ and $\lambda \in \mathbb{R}$. It is called *Legendre's differential equation*.

(a) Consider the special case $m = 0$. Make a power series ansatz for the solution, $P(\xi) = \sum_{j=1}^{\infty} a_j \xi^j$. From the differential equation derive a recursion relation between coefficients a_j and a_{j+2}. Show that the power series diverges at the endpoints $\xi = \pm 1$ **unless** $\lambda = l(l+1)$ where $l \in \mathbb{N}$ is a non-negative integer.

(b) The outcome of (a) suggests that the only physically acceptable, non-singular solutions to Legendre's equation for $m = 0$ are polynomials and they can be labeled by a quantum number l with $\lambda = l(l+1)$. Show that these *Legendre polynomials* are given by

$$P_l(\xi) = \frac{1}{2^l l!}\frac{d^l}{d\xi^l}(\xi^2 - 1)^l\,. \tag{2}$$

(The normalization is simply a convention.) What is the degree of P_l?

(c) Write down the first four Legendre polynomials ($l = 0, 1, 2, 3$) explicitly.

(d) Show that Legendre polynomials are mutually orthogonal with respect to a scalar product defined as integration over the interval $[-1, 1]$, and their norm is $\sqrt{2/(2l+1)}$, i.e.

$$\int_{-1}^{1} P_l(\xi)P_{l'}(\xi)d\xi = \frac{2}{2l+1}\delta_{ll'}\,. \tag{3}$$

(e) Now we return to the general case of Legendre's differential equation. Show that for $m \leq l$ the functions

$$P_l^m(\xi) = (1-\xi^2)^{\frac{m}{2}}\frac{d^m}{d\xi^m}P_l(\xi) \tag{4}$$

are solutions to (1) . They are called *associated Legendre functions of the first kind*.

[2] **Angular Momentum Operators** (40 points)

(a) Show the following commutation relations for the angular momentum operator $\vec{L} = \vec{r}\times\vec{p}$: (i) $[L_j, L_k] = \epsilon_{jkl} i\hbar L_l$, $j, k, l = 1, 2, 3$ where ϵ_{jkl} is the usual anti-symmetruc Levi-Civita tensor with $\epsilon_{123} = 1$; (ii) $[L_j, L^2] = 0$ for $j = 1, 2, 3$ where $L^2 = L_1^2 + L_2^2 + L_3^2$.

(b) Derive the nabla operator ∇ and the Laplace operator $\triangle$ in spherical coordinates r, θ, ϕ.

(c) Give explicit expressions of the operators L_X, L_y and L_z, in coordinate space represenations in *spherical coordinates* and show that in particular

$$L^2 = -\hbar^2 \left[\frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) \right] . \tag{5}$$

(d) Since L_z and L^2 are commuting operators we can find common eigenfunctions. Solve the two eigenvalue equations[1]

$$L_z Y(\theta, \phi) = m\hbar Y(\theta, \phi) , \tag{6}$$

$$L^2 Y(\theta, \phi) = \lambda\hbar^2 Y(\theta, \phi) \tag{7}$$

by choosing a separation ansatz $Y(\theta, \phi) = \Phi(\phi)\Theta(\theta)$. The functions $Y(\theta, \phi)$ in proper normalization (discussed later) are called *spherical harmonics*.
Hint: First solve for $\Phi(\phi)$ (what are the allowed values for m?) and then show that the equation for Θ reduces to Legendre's differential equation from problem [1].

[3] **Ground State Splitting for the Double Harmonic Osciallator** (20 points)

Consider a particle of mass m in a double harmonic oscillator potential $V(x) = \frac{1}{2}m\omega^2(|x| - a)^2$ where a is the parameter determining the separation of the two harmonic oscillator minima.

(a) We choose trial functions

$$\psi_\pm^n = N_\pm^n \left[\psi_n(x - a) \pm \psi_n(x + a)\right] \tag{8}$$

as discussed in section III.4, where the ψ_n are the usual harmonic oscillator eigenfunctions. Calculate the values of the functional $\langle H \rangle[\psi_\pm^0]$ for the case $n = 0$. As you know they are approximations to the energies of the true ground state and first excited state.

(b) In the asymptotic limit $a \to \infty$ the integrals you obtained in (a) should evaluate to simple expressions. Show that the leading terms in this limit are

$$\langle H \rangle[\psi_\pm^0] = \frac{1}{2}\hbar\omega \mp \frac{\alpha}{\sqrt{\pi}} e^{-\alpha^2} \tag{9}$$

in terms of the dimensionless parameter $\alpha = \sqrt{m\omega/\hbar}x$. Thus which trial function, ψ_+^0 or ψ_-^0, is the approximation for the ground state?

[1] It is customary to write powers of $\hbar$ explicitly in this eigenvalue problem.