Physics 606 (Quantum Mechanics I) — Spring 2014

Homework 8

Instructor: Rainer J. Fries | *Turn in your work by April 3*

[1] **δ-Function Potential** (25 points)

Consider a particle of mass m subject to a δ-shaped barrier, i.e. with potential energy $V(x) = C\delta(x)$ where $C > 0$.

(a) Discuss the stationary solutions for this problem. Derive the M-matrix and the coefficients of transmission and reflexivity of the barrier for incoming plane waves.
Hint: Integrate the Schrödinger Equation in a small region around $x = 0$ to see the effect of the δ-function potential on the matching of the asymptotic solutions for $x > 0$ and $x < 0$.

(b) Obviously the δ-function barrier can be thought of as an appropriate limit of a finite barrier of width $2a$ and height V_0 as discussed in II.2 in the lecture. How are a, V_0 and C related in that limit? Show that you recover the M-matrix from part (a) if you take the correct limit of the M-matrix of the finite barrier as discussed in class.

[2] **Hamilton's Principle for Fields** (25 points)

Consider a field $\psi(x)$ as a function of coordinates $x = (x_i)_{i=1}^N$. Let $\mathcal{L}(\psi, \frac{\partial\psi}{\partial x_j}, x)$ be the *Lagrange density* for ψ, depending on ψ, its first derivatives, and the position vector x. Let

$$S[\psi] = \int_\Gamma \mathcal{L}\left(\psi, \frac{\partial\psi}{\partial x_j}, x\right) dx^N \tag{1}$$

be the action defined as an integral of the Lagrange density over a region Γ in $\mathbb{R}^N$. In the following we only consider fields ψ that take fixed values on the boundary of Γ, denoted as $\partial\Gamma$. Show that the following two statements are equivalent:[1]

(i) $\psi(x)$ is an extremum of the functional S, i.e. small variations $\delta\psi(x)$ around $\psi(x)$ consistent with the boundary conditions leave S invariant: $\delta S = 0$.

(ii) ψ satiesfies the Euler-Lagrange field equation

$$\frac{\partial\mathcal{L}}{\partial\psi} - \frac{\partial}{\partial x_j}\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial x_j}\right)} = 0. \tag{2}$$

Hint: You can parameterize small deviations from $\psi(x)$ as $\psi(x,\alpha) = \psi(x) + \alpha\eta(x)$ where α is a "small" parameter and $\eta(x)$ is a test function which has to vanish on $\partial\Gamma$. Then $\delta S = (\partial S/\partial\alpha)\delta\alpha$; OR take your favorite classical mechanics textbook, look up the derivation of

[1] This statement can be easily generalized to a Lagrange density involving several fields $\psi_i(x)$, as for example required for the complex Schrödinger field.

the Euler-Lagrange equations from the Hamilton Principle when ψ is only a function of one parameter (time in classical mechanics) and generalize it to the case of a multi-dimensional parameter space.

[3] **Triangular Potential – Exact Solution** (25 points)

Consider a particle of mass m in a linear confining potential $V(x) = b|x|$.

(a) Show that the time-independent Schrödinger equation in this case can be rewritten as a differential equation of the type

$$\frac{d^2}{dx^2}\psi - x\psi = 0\,. \tag{3}$$

The solutions to this equation are the famous Airy-functions $Ai(x)$ and $Bi(x)$ with $\lim_{x\to\infty} Ai(x) = 0$ and $\lim_{x\to\infty} Bi(x) = \infty$. If you are not familiar with Airy functions you can find basic information at
`http://mathworld.wolfram.com/AiryFunctions.html`

(b) Now you can discuss the energy eigenfunctions and eigenvalues for this potential. Give the two lowest energy eigenvalues explicitly (the zeros of Ai and its derivative Ai' with smallest absolute values are -2.33811 and -1.01879, respectively).

[4] **Triangular Potential in 1-Parameter Approximations** (25 points)

Consider again the situation of problem [3].

(a) Approximate the ground state solution by a Gaussian function of type $e^{-\alpha^2 x^2}$ with parameter α. Find the value of α that makes the functional $\langle H \rangle$ stationary. Compare the energy eigenvalue you obtain for the ground state with the true value from [3].

(b) Repeat the discussion using a Gaussian with one node of type $xe^{-\alpha^2 x^2}$ as an approximation for the first excited state. Again determine the best value for the energy eigenvalue and compare to the result of [3].

(c) Repeat (a) by using an exponential function $e^{-\beta|x|}$ for the ground state. Which trial function gives the better approximation to the ground state?