[1] Hermite Polynomials (25 points)

(a) Show that for Hermite polynomials of degree \( n \in \mathbb{N} \)
\[
\frac{d}{d\xi} H_n(\xi) = 2n H_{n-1}(\xi) \tag{1}
\]

(b) Prove that the generating function
\[
F(\xi, s) = \sum_{n=0}^{\infty} \frac{H_n(\xi)}{n!} s^n \tag{2}
\]
can be written in closed form as
\[
F(\xi, s) = e^{\xi^2 - (s-\xi)^2} \tag{3}
\]

Hint: Integrate the differential equation for \( F \) which follows from the relation in (a).

(c) Using the generating function show that
\[
H_n(\xi) = (-1)^n e^{\xi^2} \frac{d^n}{d\xi^n} e^{-\xi^2} \tag{4}
\]

(d) Prove the integral representation
\[
H_n(\xi) = \frac{2^n}{\sqrt{\pi}} \int_{\mathbb{R}} (\xi + is)^n e^{-s^2} ds. \tag{5}
\]

Hint: Induction using (a).

[2] Important Amplitudes for the Harmonic Oscillator (25 points)

(a) Consider the function
\[
I(s, t, \lambda) = \int_{\mathbb{R}} F(\xi, s) F(\xi, t) e^{-\xi^2 + 2\lambda \xi} d\xi \tag{6}
\]
which contains the generating function \( F \) for Hermite polynomials. Show with the help of problem [1] that
\[
I(s, t, \lambda) = \sqrt{\pi} e^{\lambda^2 + 2(st + \lambda s + \lambda t)}, \tag{7}
\]
and on the other hand
\[
I_{nmk} := \int_{\mathbb{R}} H_n(\xi) H_m(\xi) \xi^k e^{-\xi^2} d\xi = \frac{1}{2^k} \frac{\partial^{n+m+k} I}{\partial s^n \partial t^m \partial \lambda^k} \Big|_{s,t,\lambda=0} \tag{8}
\]
for \( n, m, k \in \mathbb{N} \). Thus \( I \) is a generating function for integrals of the type \( I_{nmk} \).

Hint: We have discussed the special case \( k = 0 \) in class.
(b) Consider a particle of mass $m$ in a harmonic oscillator potential $\frac{1}{2}m\omega^2x^2$. Use the results from (a) to prove that for stationary states $\psi_n(x)$

$$\langle \psi_n|x\psi_{n'} \rangle = \int_{\mathbb{R}} \psi_n(x)x\psi_{n'}(x)dx = \sqrt{\frac{\hbar}{2m\omega}} \left[ \sqrt{n} \delta_{n',n-1} + \sqrt{n+1} \delta_{n',n+1} \right].$$

(9)

for $n, n' \in \mathbb{N}$

(c) In the same situation as in (b), what is $\langle \psi_n|x^2\psi_{n'} \rangle$?


Consider a particle of mass $m$ in a harmonic oscillator potential $\frac{1}{2}m\omega^2x^2$.

(a) Use the explicit solution for the time evolution of the wave function $\psi(x,t)$ with given initial condition $\psi(x,0)$ at $t = 0$ to show that the expectation values of position and momentum of the particle follow a classical motion, i.e.

$$\langle x \rangle(t) = \langle x \rangle(0) \cos \omega t + \frac{\langle p \rangle(0)}{m\omega} \sin \omega t$$

$$\langle p \rangle(t) = m \frac{d\langle x \rangle(t)}{dt}$$

(10)

Compare to the result from HW 5, [1] which was obtained without using explicit solutions to the harmonic oscillator.

(b) Calculate $\langle \psi_n|p^2\psi_{n'} \rangle$, $n, n' \in \mathbb{N}$ for any stationary states $\psi_n$ ($p$ is the momentum operator). Use the result to validate the virial theorem for stationary states.

[4] Scattering off a 1-D Square Potential (25 points)

(a) Consider a potential barrier of height $V_0$ and width $2a$ as introduced in II.2.2 in class. Discuss energy eigenstates with energy above the barrier height, i.e. $E > V_0$. What is the general form of the energy eigenfunctions? Derive the $M$-matrix from the matching conditions and discuss the transmission and reflection coefficients $T$ and $R$.

(b) Repeat the discussion for a potential well of depth $-V_0$ and width $2a$ as in II.3 in class. Discuss unbound energy eigenstates, i.e. $E > 0$. What is the general form of the energy eigenfunctions? Derive the $M$-matrix from the matching conditions and discuss the transmission and reflection coefficients $T$ and $R$.

Hint: Find similarities between the situations in (a) and (b).