Physics 606 (Quantum Mechanics I) — Spring 2014

Homework 7

Instructor: Rainer J. Fries *Turn in your work by March 25*

[1] **Hermite Polynomials** (25 points)

(a) Show that for Hermite polynomials of degree $n \in \mathbb{N}$

$$\frac{d}{d\xi} H_n(\xi) = 2n H_{n-1}(\xi) \tag{1}$$

(b) Prove that the generating function

$$F(\xi, s) = \sum_{n=0}^{\infty} \frac{H_n(\xi)}{n!} s^n \tag{2}$$

can be written in closed form as

$$F(\xi, s) = e^{\xi^2 - (s-\xi)^2} \tag{3}$$

Hint: Integrate the differential equation for F which follows from the relation in (a).

(c) Using the generating function show that

$$H_n(\xi) = (-1)^n e^{\xi^2} \frac{d^n}{dx^n} e^{-\xi^2} \tag{4}$$

(d) Prove the integral representation

$$H_n(\xi) = \frac{2^n}{\sqrt{\pi}} \int_{\mathbb{R}} (\xi + is)^n \, e^{-s^2} ds \, . \tag{5}$$

Hint: Induction using (a).

[2] **Important Amplitudes for the Harmonic Oscillator** (25 points)

(a) Consider the function

$$I(s, t, \lambda) = \int_{\mathbb{R}} F(\xi, s) F(\xi, t) e^{-\xi^2 + 2\lambda\xi} d\xi \tag{6}$$

which contains the generating function F for Hermite polynomials. Show with the help of problem [1] that

$$I(s, t, \lambda) = \sqrt{\pi} e^{\lambda^2 + 2(st + \lambda s + \lambda t)} \, , \tag{7}$$

and on the other hand

$$I_{nmk} := \int_{\mathbb{R}} H_n(\xi) H_m(\xi) \xi^k e^{-\xi^2} d\xi = \frac{1}{2^k} \frac{\partial^{n+m+k} I}{\partial s^n \partial t^m \partial \lambda^k} \Big|_{s,t,\lambda=0} \tag{8}$$

for $n, m, k \in \mathbb{N}$. Thus I is a generating function for integrals of the type I_{nmk}.
Hint: We have discussed the special case $k = 0$ in class.

(b) Consider a particle of mass m in a harmonic oscillator potential $\frac{1}{2}m\omega^2x^2$. Use the results from (a) to prove that for stationary states $\psi_n(x)$

$$\langle\psi_n|x\psi_{n'}\rangle = \int_{\mathbb{R}} \psi_n(x)x\psi_{n'}(x)dx = \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}\,\delta_{n',n-1} + \sqrt{n+1}\,\delta_{n',n+1}\right]. \quad (9)$$

for $n, n' \in \mathbb{N}$

(c) In the same situation as in (b), what is $\langle\psi_n|x^2\psi_{n'}\rangle$?

[3] **Harmonic Oscillator: Averages and Virial Theorem** (25 points)

Consider a particle of mass m in a harmonic oscillator potential $\frac{1}{2}m\omega^2x^2$.

(a) Use the explicit solution for the time evolution of the wave function $\psi(x,t)$ with given initial condition $\psi(x,0)$ at $t = 0$ to show that the expectation values of position and momentum of the particle follow a classical motion, i.e.

$$\langle x\rangle(t) = \langle x\rangle(0)\cos\omega t + \frac{\langle p\rangle(0)}{m\omega}\sin\omega t \qquad \langle p\rangle(t) = m\frac{d\langle x\rangle(t)}{dt} \quad (10)$$

Compare to the result from HW 5, [1] which was obtained without using explicit solutions to the harmonic oscillator.

(b) Calculate $\langle\psi_n|p^2\psi_{n'}\rangle$, $n, n' \in \mathbb{N}$ for any stationary states ψ_n (p is the momentum operator). Use the result to validate the virial theorem for stationary states.

[4] **Scattering off a 1-D Square Potential** (25 points)

(a) Consider a potential barrier of height V_0 and width $2a$ as introduced in II.2.2 in class. Discuss energy eigenstates with energy above the barrier height, i.e. $E > V_0$. What is the general form of the energy eigenfunctions? Derive the M-matrix from the matching conditions and discuss the transmission and reflection coefficients T and R.

(b) Repeat the discussion for a potential well of depth $-V_0$ and width $2a$ as in II.3 in class. Discuss unbound energy eigenstates, i.e. $E > 0$. What is the general form of the energy eigenfunctions? Derive the M-matrix from the matching conditions and discuss the transmission and reflection coefficients T and R.
Hint: Find similarities between the situations in (a) and (b).