Physics 606 — Spring 2014

Homework 6

Instructor: Rainer J. Fries *Turn in your work by March 4*

[1] **Translationally Invariant Systems** (25 points)

(a) Consider the unitary operator
$$U_a = e^{\frac{i}{\hbar}pa} \tag{1}$$
for translations by a in one dimension where p is the momentum operator. Show that its eigenvalues cover the complete unit circle in $\mathbb{C}$, and that they can be parameterized by $e^{\frac{i}{\hbar}Ka}$ where the K are eigenvalues to the momentum operator p, restricted to $-\frac{h}{2a} \leq K \leq \frac{h}{2a}$. What are the corresponding eigenfunctions? What is the degeneracy of each eigenvalue? Is it countable?
This range for K is called the first Brillouin zone.

(b) Show that the space of eigenfunctions of U_a for a fixed eigenvalue (given by the momentum eigenvalue K as above) can be written in the form
$$\psi_K(x) = e^{\frac{i}{\hbar}Kx}u(x) \tag{2}$$
where u is a square-integrable, periodic function with period a, i.e. $u(x+a) = u(x)$.
Eigenfunctions of the form (2) are called Bloch functions. They play an important role in crystals and other periodic lattices.

[2] **Galilei Boosts** (25 points)

(a) Recall that a Galilei boost with velocity $\vec{w}$ acts on a wave function as
$$\psi(\vec{r},t) \mapsto e^{\frac{i}{\hbar}\left(m\vec{w}\cdot\vec{r} - \frac{1}{2}mw^2 t\right)}\psi(\vec{r} - \vec{w}t, t)\,. \tag{3}$$
Show that boosts in x-, y- and z-direction can be represented by unitary operators
$$D_{w_i} = e^{\frac{i}{\hbar}K_i w_i} \tag{4}$$
$i = 1, 2, 3$, with Hermitian generators
$$K_i = mr_i - p_i t\,. \tag{5}$$
Here $\vec{r}$ and $\vec{p}$ are the position and momentum operators for a particle of mass m and t is time.
Hint: Baker-Campbell-Hausdorff

(b) For a system with potential energy $V = 0$ compute the commutators of the boost generators K_i with the other generators of the Galilei group discussed so far:
$$[K_i, K_j]\ , \quad [K_i, p_j]\ , \quad [K_i, H] \tag{6}$$

for $i, j = 1, 2, 3$.
The set of generators with the commutators as a Lie product is called the Galilei algebra.

(c) Let $D_{\vec{w}_1}$, $D_{\vec{w}_2}$ be the unitary operators representing boosts by velocities $\vec{w}_1$, $\vec{w}_2$, respectively, and let $D_{\vec{a}}$ represent a spatial translation by $\vec{a}$. Show that $D_{\vec{w}_2} D_{\vec{w}_1} = D_{\vec{w}_2+\vec{w}_1}$, i.e. the operators from (a) establish a true (non-projective) representation of boosts alone as a subgroup of $\mathcal{G}_+^+$. Now consider a spatial translation followed by a boost, once as a produce of the individual operators $D_{\vec{w}_1} D_{\vec{a}}$, and once as the single operator $D_{\vec{w}_1 \oplus \vec{a}} = e^{\frac{i}{\hbar}(K \cdot \vec{w}_1 + \vec{p} \cdot \vec{a})}$ that represents it. From a comparison of the two conclude whether the representation of the Galilei group discussed here is projective.

[3] **Free Particles as a Limit of a Large Potential Well** (25 points)

Consider an infinitely deep potential well of size L with $V(\vec{r}) = 0$ for $-L/2 \leq r_i \leq L/2$ for $i = 1, 2, 3$, $V(\vec{r}) \to \infty$ elsewhere. Unlike in I.11.4 we now consider solutions of the Schrödinger equation for a particle of mass m in the potential $V(\vec{r})$ with periodic boundary conditions (i.e. for opposite boundary points the value of ψ and all of its derivatives coincide).
The potential well with periodic boundary conditions and size $L \to \infty$ is a useful approximation of free particles.

(a) Find the wave functions (with proper normalizations) that are simultaneous eigenfunctions for the three components of the momentum operator, p_x, p_y, p_z, together with their eigenvalues. Demonstrate that they are also energy eigenstates of the Hamilton operator and give their energy eigenvalues. Show that for $L \to \infty$ the eigenvalues and eigenstates of free particles (albeit with different normalizations) are recovered.

(b) Introduce a quantum phase space density ρ by counting the number of eigenstates in a phase space volume $V_p = L^3 \Delta p_x \Delta p_y \Delta p_z$ and dividing by V_p. What is the value of ρ? Thus what is the average phase space volume occupied by an individual eigenstate?
This is an important result for statistical quantum mechanics.

(c) Introduce a density $\sigma = \Delta N / \Delta E$ of eigenstates in the energy spectrum by counting the number of states ΔN in an energy interval ΔE. Calculate σ as a function of energy E for large E.

[4] **Particle On a Cylinder** (25 points)

Consider a particle of mass m that is constrained to move only on the mantle of a cylinder of radius R with the z-axis as its symmetry axis.

(a) Derive the nabla operator ∇ and Laplace operator Δ in cylindrical coordinates.

(b) Determine the energy eigenvalues and eigenfunctions of the particle on the cylinder.