[1] **Translationally Invariant Systems** (25 points)

(a) Consider the unitary operator

$$ U_a = e^{i \frac{\hbar}{\mp} p a} $$

for translations by $a$ in one dimension where $p$ is the momentum operator. Show that its eigenvalues cover the complete unit circle in $\mathbb{C}$, and that they can be parameterized by $e^{i K a}$ where the $K$ are eigenvalues to the momentum operator $p$, restricted to $-\frac{h}{2a} \leq K \leq \frac{h}{2a}$. What are the corresponding eigenfunctions? What is the degeneracy of each eigenvalue? Is it countable?

*This range for $K$ is called the first Brillouin zone.*

(b) Show that the space of eigenfunctions of $U_a$ for a fixed eigenvalue (given by the momentum eigenvalue $K$ as above) can be written in the form

$$ \psi_K(x) = e^{i K x} u(x) $$

where $u$ is a square-integrable, periodic function with period $a$, i.e. $u(x + a) = u(x)$.

*Eigenfunctions of the form (2) are called Bloch functions. They play an important role in crystals and other periodic lattices.*

[2] **Galilei Boosts** (25 points)

(a) Recall that a Galilei boost with velocity $\vec{w}$ acts on a wave function as

$$ \psi(\vec{r}, t) \mapsto e^{i \frac{\hbar}{\mp} (m \vec{w} \cdot \vec{r} - \frac{1}{2} m \vec{w}^2 t)} \psi(\vec{r} - \vec{w} t, t) . $$

Show that boosts in $x$-, $y$- and $z$-direction can be represented by unitary operators

$$ D_{w_i} = e^{i \frac{\hbar}{\mp} K_i w_i} $$

$i = 1, 2, 3$, with Hermitian generators

$$ K_i = m r_i - p_i t . $$

*Here $\vec{r}$ and $\vec{p}$ are the position and momentum operators for a particle of mass $m$ and $t$ is time.*

*Hint: Baker-Campbell-Hausdorff*

(b) For a system with potential energy $V = 0$ compute the commutators of the boost generators $K_i$ with the other generators of the Galilei group discussed so far:

$$ [K_i, K_j] , \quad [K_i, p_j] , \quad [K_i, H] $$
for \( i, j = 1, 2, 3 \).

The set of generators with the commutators as a Lie product is called the Galilei algebra.

(c) Let \( D_{\vec{w}_1}, D_{\vec{w}_2} \) be the unitary operators representing boosts by velocities \( \vec{w}_1, \vec{w}_2 \), respectively, and let \( D_{\vec{a}} \) represent a spatial translation by \( \vec{a} \). Show that \( D_{\vec{w}_2} D_{\vec{w}_1} = D_{\vec{w}_2 + \vec{w}_1} \), i.e. the operators from (a) establish a true (non-projective) representation of boosts alone as a subgroup of \( G_+^+ \). Now consider a spatial translation followed by a boost, once as a produce of the individual operators \( D_{\vec{w}_1} D_{\vec{a}} \), and once as the single operator \( D_{\vec{w}_1 \oplus \vec{a}} = e^{\frac{i}{\hbar}(K \cdot \vec{w}_1 + p \cdot \vec{a})} \) that represents it. From a comparison of the two conclude whether the representation of the Galilei group discussed here is projective.


Consider an infinitely deep potential well of size \( L \) with \( V(\vec{r}) = 0 \) for \( -L/2 \leq r_i \leq L/2 \) for \( i = 1, 2, 3 \), \( V(\vec{r}) \to \infty \) elsewhere. Unlike in I.11.4 we now consider solutions of the Schrödinger equation for a particle of mass \( m \) in the potential \( V(\vec{r}) \) with periodic boundary conditions (i.e. for opposite boundary points the value of \( \psi \) and all of its derivatives coincide).

The potential well with periodic boundary conditions and size \( L \to \infty \) is a useful approximation of free particles.

(a) Find the wave functions (with proper normalizations) that are simultaneous eigenfunctions for the three components of the momentum operator, \( p_x, p_y, p_z \), together with their eigenvalues. Demonstrate that they are also energy eigenstates of the Hamilton operator and give their energy eigenvalues. Show that for \( L \to \infty \) the eigenvalues and eigenstates of free particles (albeit with different normalizations) are recovered.

(b) Introduce a quantum phase space density \( \rho \) by counting the number of eigenstates in a phase space volume \( V_p = L^3 \Delta p_x \Delta p_y \Delta p_z \) and dividing by \( V_p \). What is the value of \( \rho \)? Thus what is the average phase space volume occupied by an individual eigenstate?

This is an important result for statistical quantum mechanics.

(c) Introduce a density \( \sigma = \Delta N/\Delta E \) of eigenstates in the energy spectrum by counting the number of states \( \Delta N \) in an energy interval \( \Delta E \). Calculate \( \sigma \) as a function of energy \( E \) for large \( E \).

[4] Particle On a Cylinder (25 points)

Consider a particle of mass \( m \) that is constrained to move only on the mantle of a cylinder of radius \( R \) with the \( z \)-axis as its symmetry axis.

(a) Derive the nabla operator \( \nabla \) and Laplace operator \( \Delta \) in cylindrical coordinates.

(b) Determine the energy eigenvalues and eigenfunctions of the particle on the cylinder.