Physics 606 — Spring 2014

Homework 5

Instructor: Rainer J. Fries *Turn in your work by February 25*

[1] **Expectation Values in a Harmonic Oscillator Potential** (25 points)

Consider the 1-dimensional harmonic oscillator with Hamilton operator

$$H = T + V = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 \tag{1}$$

with constants $m, \omega > 0$.

Work out the following problems without using explicit solutions of the Schrödinger equation.

(a) Show that the expectation value for the position $\langle x \rangle(t)$ as a function of time t obeys the same differential equation as the position of a classical particle with the corresponding Hamilton function. Solve the equation for given initial expectation values $\langle x \rangle_0$ and $\langle p \rangle_0$ at $t = 0$.

(b) Show that for the expectation value of the square of the position

$$\langle x^2 \rangle(t) = \frac{\langle p^2 \rangle_0}{m^2\omega^2} \sin^2 \omega t + \langle x^2 \rangle_0 \cos^2 \omega t + \frac{1}{2m\omega} \langle xp + px \rangle_0 \sin 2\omega t \tag{2}$$

holds where $\langle \ldots \rangle_0$ are expectation values at $t = 0$.

Hint: It is straight forward to first derive and solve a differential equation for $\langle T - V \rangle$, then to find the solution for $\langle V \rangle \sim \langle x^2 \rangle$.

(c) Using the results from (a) and (b) calculate the time evolution of the variance $(\Delta x)^2(t)$ of a wave packet in the harmonic oscillator. Test your result by showing that for $\omega \to 0$ you recover the result for a free particle from HW IV, [3].

[2] **Density and Current Operators** (25 points)

(a) The classical density and current for a particle of mass m undergoing a motion $\vec{R}(t)$ in $\mathbb{R}^3$ is obviously

$$\rho_{\text{cl}}(\vec{r}) = \delta(\vec{r} - \vec{R}(t)) \qquad \vec{j}_{\text{cl}}(\vec{r}) = \dot{\vec{R}} \delta(\vec{r} - \vec{R}(t)) \,. \tag{3}$$

Show that the classical density and current satisfy the continuity equation.

(b) Consider the *bi-local* operators

$$\rho(\vec{r}, \vec{R}) = \delta(\vec{r} - \vec{R}) \qquad \vec{j}(\vec{r}, \vec{R}) = \frac{1}{2m} \left(\vec{p} \delta(\vec{r} - \vec{R}) + \delta(\vec{r} - \vec{R}) \vec{p} \right) \tag{4}$$

where $\vec{r}$, $\vec{R}$ and $\vec{p}$ are now operators in coordinate space representation. Show that they are the correct quantum mechanical replacement for the classical quantities in the sense that their expectation values in a state $\psi(\vec{r})$ yield the correct quantum mechanical expressions for the density and current of the field ψ.

[3] **Schrödinger Equation with Electromagnetic Potential** (25 points)

Recall that the classical Hamilton function for a particle of mass m and charge q subject to electrostatic and magnetic potentials $\phi(\vec{r},t)$ and $\vec{A}(\vec{r},t)$, respectively, is

$$H = \frac{1}{2m}\left(\vec{p} - q\vec{A}\right)^2 + q\phi\,. \tag{5}$$

(a) What are the Hamilton-Jacobi equation for the classical action S_{cl} and the classical continuity equation in this case?
Hint: Note that in the presence of a vector potential $\vec{A}$ the relevant velocity is $(\vec{p} - q\vec{A})/m$.

(b) Show that the *Schrödinger equation with electromagnetic potentials*

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{r},t) = \frac{1}{2m}\left(-i\hbar\nabla - q\vec{A}\right)^2\psi(\vec{r},t) + q\phi\psi(\vec{r},t) \tag{6}$$

for a wave function

$$\psi = C(\vec{r},t)e^{\frac{i}{\hbar}S(\vec{r},t)} \tag{7}$$

with real amplitude C and phase S reduces to the two classical equations from (a) for $\hbar \to 0$ and $S \to S_{\text{cl}}$.

[4] **Gauge Invariance and Lorentz Force** (25 points)

(a) Show that the Schrödinger equation from problem [3](b) is invariant under the simultaneous gauge transformations

$$\vec{A} \mapsto \vec{A} + \nabla f \qquad \phi \mapsto \phi - \frac{\partial f}{\partial t} \qquad \psi \mapsto e^{\frac{i}{\hbar}qf}\psi \tag{8}$$

where $f(\vec{r},t)$ is a real-valued function.

(b) Show that the quantum mechanical Lorentz force is given by

$$m\frac{d}{dt}\langle\vec{v}\rangle = \frac{q}{2}\left\langle\vec{v}\times\vec{B} - \vec{B}\times\vec{v}\right\rangle + q\langle\vec{E}\rangle \tag{9}$$

where the velocity operator is given by the operator identity

$$\vec{v} = \frac{1}{m}\left(\vec{p} - q\vec{A}\right)\,. \tag{10}$$

and $\vec{E} = -\nabla\phi - \partial\vec{A}/\partial t$ and $\vec{B} = \nabla\times\vec{A}$ are the usual electric and magnetic fields.