Physics 606 — Spring 2014

Homework 4

Instructor: Rainer J. Fries *Turn in your work by February 18*

[1] **Commutators in Coordinate and Momentum Space** (25 points)

Let $F(\vec{r})$ and $G(\vec{p})$ be two physical quantities as a function of coordinate $\vec{r}$ and momentum $\vec{p}$ respectively. Let F_p, G_p and F_r, G_r be the operators representing F and G in momentum and coordinate space respectively, acting on spaces of sufficiently fast falling functions $\mathcal{S}_r$ and $\mathcal{S}_p$ respectively. Prove that the commutators of F and G in both representations are related by Fourier transformation, i.e.

$$[F_r, G_r]\, f(\vec{r}) = (2\pi\hbar)^{-3/2} \int_{\mathbb{R}^3} [F_p, G_p]\, \hat{f}(\vec{p}) e^{\frac{i}{\hbar}\vec{r}\cdot\vec{p}} d^3p \qquad (1)$$

for any test function $f \in \mathcal{S}_r$ with Fourier transform $\hat{f} \in \mathcal{S}_p$.

Note: This can easily be generalized to a proof of the statement in I.7.2 in the lecture which makes the same statement for arbitary F, G as long as pairs of conjugate variables are separable (i.e. do not appear in products).

[2] **Commutators and Poisson Brackets; Products of Conjugate Variables** (25 points)

Let $H = p^2/2m + V(\vec{r})$ be the Hamilton operator of a system.

(a) Calculate the commutator $[\vec{r}\cdot\vec{p}, H]$ where $\vec{r}$ and $\vec{p}$ are the position and momentum operator, respectively. Show that the commutator $[\vec{p}\cdot\vec{r}, H]$ results in the same operator.

(b) Compute $[(\vec{r}\cdot\vec{p})(\vec{r}\cdot\vec{p}), H]$ and $[(\vec{r}\cdot\vec{p})(\vec{p}\cdot\vec{r}), H]$ for the special case $V = 0$. Compare both resulting operators by applying them to a plane wave test function $e^{i(\vec{r}\cdot\vec{p}-Et)/\hbar}$.

(c) Calculate the classical Poisson bracket[1] $\{\vec{r}\cdot\vec{p}, H\}$ and compare to the result of (a). Also compare the fundamental commutators and Poisson brackets $[r_i, p_j]$ and $\{r_i, p_j\}$, $i, j = 1, 2, 3$. What general relation do those comparisons suggest between commutators of operators and Poisson brackets of their classical counterparts?

(d) For the special case $V = 0$ calculate the classical Poisson bracket $\{(\vec{r}\cdot\vec{p})^2, H\}$ and compare to the results of (b). Is the conclusion of (c) tenable?

[1] We agree to use the definition $\{f, g\} = \sum_{k=1}^{n} \left(\frac{\partial f}{\partial r_k}\frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial r_k} \right)$.

[3] **Time Evolution of Wave Packet Widths** (25 points)

Using the equation of motion for expectation values show that for a free particle of mass m in one dimension the usual variances $(\Delta p)^2 = \langle (p - \langle p \rangle)^2 \rangle$, $(\Delta x)^2 = \langle (x - \langle x \rangle)^2 \rangle$ in momentum and coordinate space have the time dependences

$$(\Delta p)^2(t) = (\Delta p)^2(0) = \text{const.}\,, \tag{2}$$

$$(\Delta x)^2(t) = (\Delta x)^2(0) + \frac{2}{m}\left[\frac{1}{2}\langle xp + px \rangle (0) - \langle x \rangle(0)\langle p \rangle(0)\right] t + \frac{(\Delta p)^2(0)}{m^2} t^2\,, \tag{3}$$

if they exist.

[4] **Once More: Gaussian Wave Packets** (25 points)

(a) Use the result of [3] to calculate $(\Delta x)^2$ and $(\Delta p)^2$ for the propagating Gaussian wave packet from HW 2, problem [1].

(b) Now calculate $(\Delta x)^2$ and $(\Delta p)^2$ directly from the explicit results of HW 2, problem [1] and compare to (a).