Physics 606 — Spring 2014

Homework 2

Instructor: Rainer J. Fries *Turn in your work by February 4*

[1] Time Evolution of a Gaussian Wave Packet (25 points)

At time $t = 0$ consider a Gaussian wave packet (cf. HW I, [3]) centered around x_0 with an average momentum k_0, i.e.

$$\psi(x,0) = \left(\sqrt{2\pi}\sigma\right)^{-1/2} e^{ik_0 x} e^{-\frac{(x-x_0)^2}{4\sigma^2}} \tag{1}$$

(a) If this packet represents a free particle of energy $E = k_0^2/2m$ calculate the time evolution $\psi(x,t)$.

(b) Using a program of your choice (Mathematica, Matlab, etc.) plot the real and imaginary parts of $\psi(x,t)$ as well as $|\psi(x,t)|^2$ as functions of x for different values of t. Choose suitable parameters to observe the spreading of the wave packet with time. What determines the "speed" with which the width of the wave packet increases?

[2] Continuity Equation (25 points)

(a) Consider a system of classical particles described by a density $\rho(\vec{r},t)$ and velocity field $\vec{v}(\vec{r},t)$. Starting from the conservation of particle number in a co-moving volume

$$\frac{d}{dt}\int_{V(t)} \rho \, d^3r = 0 \tag{2}$$

derive the continuity equation

$$\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho\vec{v}) = 0\,. \tag{3}$$

(b) Consider matter fields $\psi_1(\vec{r},t)$ and $\psi_2(\vec{r},t)$ which obey the same time-dependent Schrödinger equation. Derive the current density $\vec{j}_{12}(\vec{r},t)$ which fulfills the continuity equation

$$\frac{\partial}{\partial t}\left(\psi_1\psi_2^*\right) + \nabla \cdot \vec{j}_{12} = 0\,. \tag{4}$$

(c) Hence what is the current $\vec{j}$ that satisfies the continuity equation for the probability density $\rho = |\psi|^2$ of a single field ψ? Show that $\vec{j}$ goes towards the result from (a), i.e. $\vec{j} \to \rho\vec{v}$ in the classical limit.

[3] A Simple Hamilton-Jacobi Problem (25 points)

Consider a particle of mass m moving in one dimension with potential energy $U(x) = -bx$. Write down the Hamilton-Jabobi equation, solve for the action S and derive the motion $x(t)$ of the system with initial conditions $x(0) = x_0$, $\dot{x}(0) = v_0$.

[4] **Discontinuous Potential Energy** (25 points)

(a) Consider a surface in $\mathbb{R}^3$ at which the potential energy $V(\vec{r})$ of a particle is discontinuous. Show that the boundary conditions

$$\lim_{\epsilon\to 0^+} \psi(\vec{S}+\epsilon\hat{n},t) = \lim_{\epsilon\to 0^-} \psi(\vec{S}+\epsilon\hat{n},t) \tag{5}$$

$$\lim_{\epsilon\to 0^+} \hat{n}\cdot\nabla_\epsilon\psi(\vec{S}+\epsilon\hat{n},t) = \lim_{\epsilon\to 0^-} \hat{n}\cdot\nabla_\epsilon\psi(\vec{S}+\epsilon\hat{n},t) \tag{6}$$

for each point $\vec{S}$ on the surface and for all times t satisfy the continuity equation. $\hat{n}$ is a unit normal vector perpendicular to the surface in $\vec{S}$. In other words, both the wave function and its derivative normal to the surface have to be continuous.

(b) Consider the situation of particles of energy E with a constant potential energy V_1 in the left half-space ($z < 0$) and another constant potential energy V_2 in the right half-space ($z > 0$). Assuming $E > V_1, V_2$ show that plane waves $\psi_{\vec{k}_1}$, $\psi_{\vec{k}_2}$ with arbitrary wave vectors $\vec{k}_1$ and $\vec{k}_2$ are solutions to the Schrödinger equation in both half-spaces *separately*. Using only the first boundary condition in (a), how are the wave vectors $\vec{k}_1$ and $\vec{k}_2$ of the two half-space plane wave solutions related? Formulate the law of refraction for matter waves with an appropriate index of refraction.

(c) Is a solution that "stitches" together two half-space plane waves $\psi_{\vec{k}_1}$ and $\psi_{\vec{k}_2}$ using the result from (b)

$$\psi_{\vec{k}_1}\,\theta(-z) + \psi_{\vec{k}_2}\,\theta(z) \tag{7}$$

actually an allowed solution of the Schrödinger equation for all of $\mathbb{R}^3$? *Hint: Check the second boundary condition.* Taking a hint from optics, how can you resolve the situation? Give an ansatz for a full solution. (You don't have to solve the equations you get from the second boundary condition.)