Physics 606 (Quantum Mechanics I) — Spring 2014

Homework 10

Instructor: Rainer J. Fries *Turn in your work by April 22*

[1] **Some More Commutators of Angular Momentum Operators** (20 points)

(a) Compute the commutators $[L_i, r_j]$, $[L_i, p_j]$ and $[L_i, K_j]$, $i = 1, 2, 3$, where L_i, r_i, p_i and K_i are the orbital angular momentum, position, momentum, and boost operators, respectively.

(b) For the raising and lowering operators $L_\pm = L_x \pm iL_y$ compute the commutators $[L_\pm, L_z]$ and $[L_+, L_-]$. Show that $L^2 - L_z^2 = L_\pm L_\mp \mp \hbar J_z$

[2] **Harmonic Oscillator Algebra** (40 points)

(a) Calculate the matrix representation of the lowering and raising operators a and $a^\dagger$ of the harmonic oscillator with respect to the energy eigenstate basis $|n\rangle$, $n \in \mathbb{N}$. I.e. calculate the matrix elements $\langle n'|a|n\rangle$, etc.

(b) With the help of (a) determine the normalization factors C_n, D_n, N_n in the following equations from lecture:

$$a|n\rangle = C_n|n-1\rangle \qquad a^\dagger|n\rangle = D_n|n+1\rangle \qquad |n\rangle = N_n(a^\dagger)^n|0\rangle \tag{1}$$

(c) Compute the matrix representation of the position operator $\hat{x}$ with respect to the basis $|n\rangle$, $n \in \mathbb{N}$.[1]

(d) Consider the trivial eigenvalue equation $\hat{x}|x\rangle = x|x\rangle$ where $\hat{x}$ is the position operator, and x the position described by the eigenstate $|x\rangle$. Derive the corresponding "matrix equation" that is the eigenvalue equation for the amplitudes $\psi_n(x) = \langle n|x\rangle$ of the positions in the basis $|n\rangle$, $n \in \mathbb{N}$.
Hint: Insert a complete set of states.

(e) Of course the $\psi_n(x)$ are just the complex conjugates of the coordinate space wave functions of the harmonic oscillator, $\langle x|n\rangle$. Show that the eigenvalue problem for the $\psi_n(x)$ in (d) is solved by

$$\psi_n(x) = 2^{-\frac{n}{2}}(n!)^{-\frac{1}{2}} h_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)\psi_0(x) \tag{2}$$

where the $h_n(x)$ satisfy the recurrence relation

$$h_{n+1}(x) - 2xh_n(x) + 2nH_{n-1}(x) = 0\,. \tag{3}$$

[1] We use the $\hat{}$ symbol for the operator x here to avoid confusion with x labeling the eigenvalues of this operator.

(f) Show that the Hermite polynomials satisfy the recurrence relations from (e), so that $h_n(x) = C(x)H_n(x)$.
Hint: You can, e.g., use the relation you proved in HW VII, [1](c).

Note: You can get the remaining unknowns $C(x)\psi_0(x)$ which contain the Gaussian and the normalization factor from the closure relation, but you don't need to do that here.

[3] **Free Particle Solutions in Spherical Coordinates** (40 points)

Consider a free particle of mass m. Since L^2 and L_z commute with the free Hamilton operator H, you can find simultaneous eigenfunctions $\psi(\vec{r}) = \psi(r, \theta, \phi)$ for H, L^2 and L_z.

(a) The ansatz $\psi_{l,m}(r, \theta, \phi) = CR(r)Y_l^m(\theta, \phi)$, where the $Y_l^m(\theta, \phi)$ are the spherical harmonics from HW IX, [2], obviously provides eigenfunctions of L^2 and L_z with eigenvalues $\hbar^2 l(l+1)$ and $\hbar m$. Show that the "radial" differential equation that $R(r)$ needs to satisfy to make $\psi_{l,m}$ an eigenfunction of H as well is

$$\frac{d^2R}{d\rho^2} + \frac{2}{\rho}\frac{dR}{d\rho} + \left[1 - \frac{l(l+1)}{\rho^2}\right] R = 0 \tag{4}$$

where we have introduced the dimensionless radial coordinate $\rho = r\sqrt{2mE}/\hbar$.

(b) The radial equation above is also called the spherical Bessel equation. Its regular solutions are the spherical Bessel functions j_l which are in integral representation defined as

$$j_l(\rho) = \frac{\rho^l}{2^{l+1}l!}\int_{-1}^{1} e^{i\rho s}(1-s^2)^l ds\,. \tag{5}$$

(We will not be interested in its singular solutions which are the spherical Neumann functions n_l). Show that the j_l given above satisfy the radial equation from (a).

(c) Show that

$$j_l(z) = (-1)^l \rho^l \left(\frac{d}{\rho d\rho}\right)^l \frac{\sin\rho}{\rho}\,. \tag{6}$$

Give the first 3 spherical Bessel functions ($l = 0, 1, 2$) explicitly and graph them.

(d) We already know that plane waves are eigenfunctions of H that are also simultaneously eigenfunctions for the components of the momentum vector $\vec{p}$. Assume a particle in a stationary plane wave $e^{i\vec{k}\cdot\vec{r}}$ with wave vector $\vec{k} = k\hat{e}_z$ pointing in z-direction. Derive an expression that describes this plane wave in terms of angular momentum eigenstates.