Physics 606 — Spring 2014

Homework 1

Instructor: Rainer J. Fries *Turn in your work by January 28*

[1] **Bohr-Sommerfeld Quantization** (25 points)

Using the quantization criterion given by Bohr and Sommerfeld calculate the possible energies allowed for a particle of mass m

(a) in an harmonic oscillator potential $U(x) = \frac{1}{2}kx^2$.

(b) in an infinitely deep 1-D potential well of width L, i.e. $U(x) = 0$ for $-L/2 < x < L/2$ and $U(x) \to \infty$ everywhere else.

[2] **Classical Waves: A Simple Example** (25 points)

Consider a string of length L and linear mass density ρ. The string is stretched out along the x-axis and fixed at both ends. The tension in the string is T. We only consider *small* displacements from mechanical equilibrium in y-direction (transverse to the string).

(a) By considering the forces on small segments of length Δx derive the wave equation for the transverse displacement $y(x,t)$ of the string from equilibrium ($y = 0$). (For small displacements you can neglect longitudinal displacements of string segments.)

(b) Find the set of solutions of the form

$$y(x,t) = w(x)h(t) \tag{1}$$

(standing waves) consistent with the boundary conditions. Which wave lengths are allowed?

(c) Now assume that you can associate a non-relativistic particle of mass m with the standing wave. Using the de Broglie relation $p = \hbar k$, what is the energy spectrum of the particle? Compare to the solution of problem [1](b).

[3] **Gaussians** (25 points)

Consider the real-valued 1-D Gauss function

$$f(x) = Ce^{-\frac{(x-x_0)^2}{4\sigma^2}} \,. \tag{2}$$

with real parameters x_0 and σ.
Note: All integrals in this problem can be done with basic real and complex calculus. Attempt to solve them yourself to receive full credit.

(a) How does one have to choose the normalization factor C such that the Gaussian has L^2-norm 1, i.e.

$$\int_{\mathbb{R}} |f(x)|^2 dx = 1\,? \tag{3}$$

(b) Calculate the expectation value of position and the variance with f^2 as the weight, i.e.

$$\langle x \rangle = \int_{\mathbb{R}} x|f(x)|^2 dx\,, \tag{4}$$

$$(\Delta x)^2 = \langle (x - x_0)^2 \rangle = \int_{\mathbb{R}} (x - x_0)^2 \, |f(x)|^2 dx\,. \tag{5}$$

What are thus the interpretations of x_0 and σ in f?

(c) Calculate the Fourier transform $f(k)$ of the Gauss function. What are the average value and variance? Show that Gaussian wave packets exhaust the inequality of the uncertainty relation, i.e. they have a minimal uncertainty

$$\Delta x \Delta k = \frac{1}{2} \tag{6}$$

[4] **Rectangular Pulse** (25 points)

Consider a wave packet shaped as a rectangular pulse in momentum space, i.e. $\phi(k) = C$ for $k_0 - a/2 < k < k_0 + a/2$ and $\phi(k) = 0$ elsewhere. Here the average momentum k_0 and width a are parameters.

(a) What is the value of the constant C to make the L^2-norm of ϕ equal to 1? Calculate the Fourier transform $\psi(x)$ of ϕ. Plot or sketch $\psi(x)$ and $|\psi(x)|^2$ for some values of a and k_0.

(b) Check the validity of Plancherel's theorem for this wave function.

(c) Calculate the variances $(\Delta x)^2$ and $(\Delta k)^2$ around $\bar{x} = 0$ and $\bar{k} = k_0$ respectively. What do you notice for Δx?

(d) Propose another suitable measure of the width of $\psi(x)$ and check whether the uncertainty relation holds for it.